

Determination of sizes of optimal three-dimensional optical orthogonal codes of weight three with the AM-OPP restriction*

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Abstract: In this paper, we further investigate the constructions on three-dimensional $(u \times v \times w, k, 1)$ optical orthogonal codes with the at most one optical pulse per wavelength/time plane restriction (briefly AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs) by way of the corresponding designs. Several new auxiliary designs such as incomplete holey group divisible designs and incomplete group divisible packings are introduced and therefore new constructions are presented. As a consequence, the exact number of codewords of an optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOC is finally determined for any positive integers v, w and $u \geq 3$.

Keywords: three-dimensional, optical orthogonal code, group divisible packing, incomplete group divisible packing, holey group divisible design

1 Introduction

Optical code-division multiple access (OCDMA) is one of the attractive multiple-access schemes for optical networks. Good features of OCDMA include accommodation of burst traffic, asynchronous transmission, etc. Optical orthogonal codes (OOCs) have been designed for OCDMA. A one-dimensional (1-D) optical orthogonal code (1-D OOC) is a set of binary sequences having good auto and cross-correlations. 1-D OOCs were first suggested by [7] in 1989. Since then there are many researches on 1-D OOCs (see, e.g., [2, 3, 5, 12, 14, 18, 19, 26]). 1-D OOCs spread optical pulses only in time domain. One limitation of 1-D OOCs is that the length of the sequence increases rapidly when the number of users or the weight of the code is increased, which means large bandwidth expansion is required if a big number of codewords is needed. To lessen this problem, two-dimensional (2-D) optical orthogonal codes (2-D OOCs) are proposed [25]. Optical pulses in 2-D OOCs are spread in both time and wavelength domain. The performance of the OCDMA system is much improved. There is a considerable literature on 2-D OOC constructions. We refer the readers to the survey [20] for more details. To further improve the performance of 2-D OOCs, three-dimensional (3-D) optical orthogonal codes (3-D OOCs) are introduced [17]. In 3-D OOCs, optical pulses are spread in space, time and wavelength domain. We define a 3-D OOC formally as follows.

*Supported by the NSFC under Grant 11401582 and the NSFHB under Grant A2015507019 (L. Wang), and the NSFC under Grants 11271042 and 11431003 (Y. Chang)

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Let u, v, w, k and λ be positive integers, where u, v and w are the number of spatial channels, wavelengths, and time slots, respectively. A *three-dimensional* $(u \times v \times w, k, \lambda)$ *optical orthogonal code* (briefly 3-D $(u \times v \times w, k, \lambda)$ -OOC), \mathcal{C} , is a family of $u \times v \times w$ $(0, 1)$ arrays (called *codewords*) of Hamming weight k satisfying: for any two arrays $A = [a(i, j, l)]$, $B = [b(i, j, l)] \in \mathcal{C}$ and any integer τ :

$$\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \sum_{l=0}^{w-1} a(i, j, l) b(i, j, l + \tau) \leq \lambda,$$

where either $A \neq B$ or $\tau \not\equiv 0 \pmod{w}$, and the arithmetic $l + \tau$ is reduced modulo w .

A wavelength/time plane is also called a *spatial plane*. There are several classes of 3-D OOCs of interests. The following two additional restrictions on the placement of pulses are often placed within the codewords of a 3-D OOC:

- the one-pulse per plane (OPP) restriction: each codeword contains exactly one optical pulse per spatial plane.
- the at-most one-pulse per plane (AM-OPP) restriction: each codeword contains at most one optical pulse per spatial plane.

It is clear that 3-D OOCs which satisfy the OPP restriction must satisfy the AM-OPP restriction. In what follows, a 3-D $(u \times v \times w, k, \lambda)$ -OOC with the AM-OPP restriction is denoted by an AM-OPP 3-D $(u \times v \times w, k, \lambda)$ -OOC.

The number of codewords of a 3-D OOC is called its *size*. Let $\Phi(u \times v \times w, k, \lambda)$ denote the largest possible size of an AM-OPP 3-D $(u \times v \times w, k, \lambda)$ -OOC. Based on the Johnson bound [16] for constant weight codes, an upper bound of $\Phi(u \times v \times w, k, \lambda)$ is given by Shum [21].

Lemma 1.1 [21] $\Phi(u \times v \times w, k, \lambda) \leq J(u \times v \times w, k, \lambda)$ holds for any positive integers $u \geq k > \lambda$, where

$$J(u \times v \times w, k, \lambda) = \lfloor \frac{uv}{k} \lfloor \frac{vw(u-1)}{k-1} \cdots \lfloor \frac{vw(u-\lambda)}{k-\lambda} \rfloor \cdots \rfloor \rfloor.$$

An AM-OPP 3-D $(u \times v \times w, k, \lambda)$ -OOC with $\Phi(u \times v \times w, k, \lambda)$ codewords is said to be *optimal*. Furthermore, we say that an optimal AM-OPP 3-D $(u \times v \times w, k, \lambda)$ -OOC is *perfect* if

$$\Phi(u \times v \times w, k, \lambda) = v^{\lambda+1} w^{\lambda} \frac{u(u-1) \cdots (u-\lambda)}{k(k-1) \cdots (k-\lambda)}.$$

Some work has been done for optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOCs. Shum [21] has solved the existence problem of perfect AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOCs.

Theorem 1.2 [21] *A perfect AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOC exists if and only if*

- (1) $u = 3$: w is odd, or w is even and v is even;
- (2) $u \geq 4$: $(u-1)vw \equiv 0 \pmod{2}$, $u(u-1)vw \equiv 0 \pmod{3}$, and $v \equiv 0 \pmod{2}$ when $u \equiv 2, 3 \pmod{4}$ and $w \equiv 2 \pmod{4}$.

The authors [24] studied the constructions for general optimal AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs. We improved the upper bound on $\Phi(u \times v \times w, 3, 1)$ and determined the size of optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOCs with some possible exceptions.

Lemma 1.3 [24] $\Phi(u \times v \times w, 3, 1) \leq J^*(u \times v \times w, 3, 1)$, where

$$J^*(u \times v \times w, 3, 1) = \begin{cases} J(u \times v \times w, 3, 1) - 1, & \text{if } v \equiv 1 \pmod{2}, w \equiv 0 \pmod{4}, \\ & \text{and } u = 3, \\ & \text{or } v \equiv 1 \pmod{2}, w \equiv 2 \pmod{4}, \\ & \text{and } u \equiv 3, 6, 7, 10 \pmod{12}, \\ & \text{or } vw \equiv 6 \pmod{12}, w \equiv 2 \pmod{4}, \\ & \text{and } u \equiv 2, 11 \pmod{12}, \\ & \text{or } (u-1)v^2w \equiv 4 \pmod{6}, \\ & \text{and } u \equiv 2 \pmod{3}, \\ & \text{or } w = 2, u(u-1)v^2 \equiv 8 \pmod{12}, \\ J(u \times v \times w, 3, 1), & \text{otherwise.} \end{cases}$$

Theorem 1.4 [4, 24, 27] *An optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOC with $J^*(u \times v \times w, 3, 1)$ codewords exists if either $w = 1$, $v \geq 1$ and $u \geq 3$, or $w > 1$, $v \geq 1$, $u \geq 3$ and $u \not\equiv 2 \pmod{6}$, except possibly for $u \equiv 11 \pmod{12}$, $v \equiv 1, 5 \pmod{6}$ and $w \equiv 10 \pmod{12}$.*

In this paper, we further study the combinatorial constructions for general optimal AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs. We introduce some new auxiliary designs and new constructions. As a consequence, we determine the sizes of optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOCs for any positive integers v, w and $u \geq 3$.

The rest of this paper is organized as follows. In Section II, we give a combinatorial description of AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs. In Sections III and IV, we introduce three types of auxiliary designs and their existence results respectively, which will be used in the constructions of AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs. In Section V, we obtain the main result of this paper. Section VI gives a concluding remark. Note that in order to save space, we do not list most of the data used in this paper. The interested reader may access them in the Supporting Information¹.

2 Combinatorial description

Optimal AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs are closely related to combinatorial designs. In this section, we define some terminologies and notations in design theory and state the link between AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOCs and the designs. We always assume that $I_n = \{0, 1, \dots, n-1\}$ and denote by Z_n the additive group of integers modulo n throughout this paper.

Let K be a set of positive integers. A *group divisible packing* (K -GDP) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfying the following properties:

- (1) X is a finite set of *points*;
- (2) \mathcal{G} is a partition of X into subsets (called *groups*);
- (3) \mathcal{B} is a collection of subsets of X (called *blocks*), each of size from K , such that any pair of X appears in a group or in at most one block, but not both.

¹Supporting Information, arXiv:

If \mathcal{G} contains u_i groups of size g_i , $1 \leq i \leq r$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ the *group type* (or *type*) of the GDP. If $K = \{k\}$, we further omit the braces to simply write k for K . If every admissible pair appears in exactly one block, this GDP is also referred to as a *group divisible design* and denoted by a K -GDD. A K -GDD of type 1^u is also called a *pairwise balance design*, or $\text{PBD}(u, K)$ for short.

We record some results concerning GDDs and PBDs, which will be used later.

Lemma 2.1 [6] *Let g, t and w be positive integers. Then there exists a 3-GDD of type $g^t w^1$ if and only if the following conditions are all satisfied: (i) $t \geq 3$, or $t = 2$ and $w = g$; (ii) $w \leq g(t - 1)$; (iii) $g(t - 1) + w \equiv 0 \pmod{2}$; (iv) $gt \equiv 0 \pmod{2}$; and (v) $g^2 t(t - 1) + 2gtw \equiv 0 \pmod{6}$.*

Lemma 2.2 [1] *A $\text{PBD}(u, \{4, 5, 6\})$ exists for any positive integer $u \geq 4$ and $u \notin \{7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$.*

Suppose $(X, \mathcal{G}, \mathcal{B})$ is a GDD. If there exists a partition $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s\}$ of \mathcal{B} such that each \mathcal{P}_i (called a *parallel class*) is a partition of X for $1 \leq i \leq s$, then this GDD is said to be *resolvable* and denoted by an RGDD. If the set of block size of a GDD (PBD) is denoted by $K \cup \{k^*\}$, we means that this GDD (PBD) has exactly one block of size k .

Now we use an RGDD to construct a PBD which has exactly one block of size 5.

Lemma 2.3 *There exists a $\text{PBD}(u, \{3, 4, 5^*\})$ for any positive integer $u \equiv 2 \pmod{6}$ and $u \geq 14$.*

Proof A $\text{PBD}(14, \{3, 4, 5^*\})$ exists by Lemma 5.3 of [15]. When $u \geq 20$, a 3-RGDD of type $3^{(u-5)/3}$ can be found in [13], which contains $(u - 8)/2$ parallel classes. Adjoining 5 infinite points to complete five parallel classes, then we obtain a $\{3, 4\}$ -GDD of type $3^{(u-5)/3} 5^1$, which is also a $\text{PBD}(u, \{3, 4, 5^*\})$. \square

Next we introduce the concept of w -cyclic GDPs, which are very useful in the constructions of AM-OPP 3-D OOCs.

Let w be a common divisor of g_i , say $g_i = v_i w$, $1 \leq i \leq r$. Suppose $(X, \mathcal{G}, \mathcal{B})$ is a K -GDP of type $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$. If there is a permutation π on X which is the product of $\sum_{i=1}^r v_i u_i$ disjoint w -cycles, fixes every group, and leaves \mathcal{B} invariant, then this design is said to be *w-cyclic*. A w -cyclic k -GDP (GDD) of type w^u is called a *semi-cyclic k-GDP* (GDD) of type w^u and denoted by a k -SCGDP (SCGDD) of type w^u .

Without loss of generality, for a w -cyclic K -GDP of type $(vw)^u$, we always identify $X = I_u \times I_v \times Z_w$, $\mathcal{G} = \{\{i\} \times I_v \times Z_w : i \in I_u\}$ and the permutation $\pi : (i, j, x) \mapsto (i, j, x + 1) \pmod{(-, -, w)}$.

Any permutation π partitions \mathcal{B} into equivalence classes called the *block orbits* under π . A set of base blocks is an arbitrary set of representatives for these block orbits of \mathcal{B} . A w -cyclic k -GDP of type $(vw)^u$ is called *optimal* if it contains the largest possible number of base blocks.

Obviously, the following lemma holds. We omit its proof.

Lemma 2.4 *If there exists a k -SCGDP of type $(vw)^u$ with b base blocks, then there exists a w -cyclic k -GDP of type $(vw)^u$ with vb base blocks.*

The authors [24] established the equivalence between an optimal AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOC and an optimal w -cyclic k -GDP of type $(vw)^u$.

Theorem 2.5 [24] *An optimal AM-OPP 3-D $(u \times v \times w, k, 1)$ -OOC is equivalent to an optimal w -cyclic k -GDP of type $(vw)^u$.*

Combining with Theorems 1.2 and 1.4, we have the following result.

Corollary 2.6 (1) *A w -cyclic 3-GDD of type $(vw)^u$ exists if and only if the conditions given in Theorem 1.2 are satisfied.*

(2) *An optimal w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks exists for the conditions shown in Theorem 1.4.*

3 Holey group divisible designs

In this section, we introduce our first auxiliary design called a *holey group divisible design* (HGDD), which will play an important role in constructions of optimal w -cyclic 3-GDPs.

A k -HGDD is defined to be a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

- (1) X is an $(u \sum_{i=1}^t g_i)$ -set (of *points*);
- (2) $\mathcal{G} = \{G_1, \dots, G_u\}$ is a partition of X into u subsets (called *groups*) of points $\sum_{i=1}^t g_i$ each;
- (3) $\mathcal{H} = \{H_1, \dots, H_t\}$ is another partition of X into t subsets (called *holes*) of points ug_i each such that $|H_i \cap G_j| = g_i$ for any $1 \leq i \leq t$ and $1 \leq j \leq u$;
- (4) \mathcal{B} is a collection of k -subsets of X (called *blocks*) such that any pair of X from two distinct groups appears in a hole or exactly in one block but not both, and no other pairs of X occur in any block.

If \mathcal{H} contains t_i holes of size ug_i , $1 \leq i \leq r$, we then use an “exponential” notation $g_1^{t_1} \dots g_r^{t_r}$ to denote the multiset $T = \{g_j : j = 1, 2, \dots, t\}$ and call (u, T) the type of the design.

The necessary and sufficient conditions on the existence of 3-HGDDs of type $(u, g^t w^1)$ have been determined by Wang and Yin [23].

Theorem 3.1 [23] *Let u, t, g and w be nonnegative integers. The necessary and sufficient conditions for the existence of a 3-HGDD of type $(u, g^t w^1)$ are that $u \geq 3$, $t = 2$ and $g = w$; or $t \geq 3$, $0 \leq w \leq g(t-1)$, $gt(u-1) \equiv 0 \pmod{2}$, $(u-1)(w-g) \equiv 0 \pmod{2}$ and $gtu(u-1)(g(t-1)-w) \equiv 0 \pmod{3}$.*

For the purpose of constructing optimal w -cyclic k -GDP of type $(vw)^u$, the HGDDs must admit some special permutations.

Suppose $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is a k -HGDD of type $(u, (gh)^{mt})$. If there is a permutation π on X which is the product of uhm disjoint gt -cycles, fixes every group, leaves \mathcal{B} invariant, and partitions the mt elements of \mathcal{H} into m equivalence classes, then this design is said to be (h, gt, m) -cyclic.

Without loss of generality, we always identify $X = I_u \times I_h \times I_m \times Z_{gt}$, $\mathcal{G} = \{\{i\} \times I_h \times I_m \times Z_{gt} : i \in I_u\}$ and $\mathcal{H} = \{I_u \times I_h \times \{i\} \times \{0+j, t+j, \dots, (g-1)t+j\} : (i, j) \in$

$I_m \times Z_t\}$. In this case, the permutation π can be taken as $(x, y, z, w) \mapsto (x, y, z, w + 1) \pmod{-, -, -, gt}$.

A $(1, gt, 1)$ -cyclic k -HGDD of type (u, g^t) is also referred to as a *semi-cyclic k -HGDD* of type (u, g^t) and denoted by k -SCHGDD of type (u, g^t) . Clearly, under the action of π , \mathcal{B} can be partitioned into equivalence classes called the *block orbits*. A set of base blocks is a set of representatives for these block orbits of \mathcal{B} .

Example 3.2 *There exists a $(3, 3, 1)$ -cyclic 3-HGDD of type $(5, 3^3)$.*

Proof Let $X = Z_5 \times Z_3 \times Z_3$, $\mathcal{G} = \{\{i\} \times Z_3 \times Z_3 : i \in Z_5\}$, and $\mathcal{H} = \{Z_5 \times Z_3 \times \{j\} : j \in Z_3\}$. Developing the following 4 initial base blocks by $(+i, +j, -) \pmod{(5, 3, -)}$ yields all 60 base blocks of the required design, where $(i, j) \in Z_5 \times Z_3$.

$$\begin{aligned} &\{(0, 0, 0), (1, 0, 1), (2, 1, 2)\}, \{(0, 0, 0), (1, 0, 2), (2, 2, 1)\}, \\ &\{(0, 0, 0), (1, 2, 1), (3, 0, 2)\}, \{(0, 0, 0), (1, 1, 2), (3, 1, 1)\}. \end{aligned} \quad \square$$

We give our first “Filling Construction” via $(h, gt, 1)$ -cyclic k -HGDDs.

Construction 3.3 [24](Filling Construction-I) *Suppose that the following designs exist:*

- (1) *an $(h, gt, 1)$ -cyclic k -HGDD of type $(u, (gh)^t)$ with b base blocks;*
- (2) *a g -cyclic k -GDP of type $(gh)^u$ with f base blocks.*

Then, there exists a gt -cyclic k -GDP of type $(ght)^u$ with $b + f$ base blocks.

For applying Construction 3.3, we need some (h, gt, m) -cyclic k -HGDDs. Below, we investigate its constructive methods and apply these constructions to obtain some existence results. Constructions 3.4 and 3.6 can be found in [24].

Construction 3.4 *If there exists a k -SCHGDD of type $(u, (gh)^t)$, then there exists an (h, g, t) -cyclic k -HGDD of type $(u, (gh)^t)$.*

Construction 3.4 indicates that k -SCHGDDs are useful in the construction of (h, g, t) -cyclic k -HGDDs. We quote the result on 3-SCHGDDs for later use.

Theorem 3.5 [9, 11] *There exists a 3-SCHGDD of type (u, g^t) if and only if $u, t \geq 3$, $(t - 1)(u - 1)g \equiv 0 \pmod{2}$, and $(t - 1)u(u - 1)g \equiv 0 \pmod{6}$ except when*

- (1) $u \equiv 3, 7 \pmod{12}$, $g \equiv 1 \pmod{2}$ and $t \equiv 2 \pmod{4}$; (2) $u = 3$, $g \equiv 1 \pmod{2}$ and $t \equiv 0 \pmod{2}$; (3) $u = t = 3$, $g \equiv 0 \pmod{2}$; (4) $(u, g, t) \in \{(5, 1, 4), (6, 1, 3)\}$;
- and possibly when*

- (1) $t = 8$, either $g \equiv 1 \pmod{2}$ and $u \equiv 1, 3 \pmod{6}$ and $u \geq 7$, or $g \equiv 3 \pmod{6}$ and $u \equiv 5 \pmod{6}$; (2) $u \equiv 1, 9 \pmod{12}$, $g \equiv 1 \pmod{2}$ and $t \equiv 2 \pmod{4}$; (3) $u \equiv 5 \pmod{6}$ and $u \geq 11$, either $g \equiv 3 \pmod{6}$ and $t \equiv 2 \pmod{4}$, or $g \equiv 1, 5 \pmod{6}$ and $t \equiv 10 \pmod{12}$.

Proof Reference [9] deals with the case when $u = 8$, $g \equiv 2, 10 \pmod{12}$ and $t \equiv 7, 10 \pmod{12}$. Other cases can be found in [11]. \square

Construction 3.6 (Inflation-I) *Suppose that there exists an (h, gt, m) -cyclic k -HGDD of type $(u, (gh)^{mt})$. If there exists a w -cyclic l -GDD of type $(vw)^k$, then there exists an (hv, gtw, m) -cyclic l -HGDD of type $(u, (ghvw)^{mt})$.*

Constructions 3.7 and 3.8 are simple generalizations of Constructions 3.2 and 3.1 of [10], respectively. It is a routine matter of checking their correctness.

Construction 3.7 *Suppose there exists a $PBD(u, K)$. If there exists an (h, gt, m) -cyclic l -HGDD of type $(k, (gh)^{mt})$ for any $k \in K$, then there exists an (h, gt, m) -cyclic l -HGDD of type $(u, (gh)^{mt})$.*

Construction 3.8 *If there exist an $(h, gt, 1)$ -cyclic k -HGDD of type $(u, (gh)^t)$ and an $(h, g, 1)$ -cyclic k -HGDD (u, h^g) , then there exists an $(h, gt, 1)$ -cyclic k -HGDD of type (u, h^{gt}) .*

Now we are ready to apply these constructions to obtain some infinite classes of (h, gt, m) -cyclic 3-HGDDs, which are essential to our work.

Lemma 3.9 *Suppose $u \equiv 0, 1 \pmod{3}$ and $u \geq 3$. Let g be a positive integer. Then there exists a $(g, t, 1)$ -cyclic 3-HGDD of type (u, g^t) for any $t \equiv 1 \pmod{2}$ and $t \geq 3$, except for $(u, g, t) = (6, 1, 3)$.*

Proof By Theorem 3.5, there is a 3-SCHGDD of type $(u, 1^t)$, which is also a $(1, t, 1)$ -cyclic 3-HGDD of type $(u, 1^t)$. Inflate it by a 1-cyclic 3-GDD of type g^3 from Corollary 2.6. Hence, by Construction 3.6, we obtain a $(g, t, 1)$ -cyclic 3-HGDD of type (u, g^t) . \square

Lemma 3.10 *Suppose $u \equiv 2 \pmod{6}$ and $u \geq 8$. Let h be a positive integer. Then there exists an $(h, gt, 1)$ -cyclic 3-HGDD of type $(u, (gh)^t)$ if g, t satisfy one of following conditions.*

- (1) $g = 1, t \equiv 1 \pmod{6}$ and $t \geq 7$; (2) $g = 2, t \equiv 1 \pmod{3}$ and $t \geq 4$;
- (3) $g = 3, t \equiv 1 \pmod{2}$ and $t \geq 3$; (4) $g = 4, t \equiv 1 \pmod{3}$ and $t \geq 4$;
- (5) $g = 6, t \equiv 1 \pmod{2}$ and $t \geq 3$.

Proof The proof is similar to that of Lemma 3.9. The required 3-SCHGDDs of type (u, g^t) and 1-cyclic 3-GDDs of type h^3 can be found in Theorem 3.5 and Corollary 2.6 respectively. \square

Lemma 3.11 *Suppose $u \equiv 2 \pmod{6}$ and $u \geq 8$. Let $g \in \{1, 2\}$. Then there exists an $(h, gt, 1)$ -cyclic 3-HGDD of type $(u, (gh)^t)$ for any $h \equiv 3 \pmod{6}$, $t \equiv 1, 5 \pmod{6}$ and $t \geq 5$.*

Proof When $h = 3$, by Theorem 3.5, there exists a 3-SCHGDD of type $(u, (3g)^t)$. Since $(3, gt) = 1$, Z_{3gt} is isomorphic to $Z_3 \times Z_{gt}$. Hence, we have a $(3, gt, 1)$ -cyclic 3-HGDD of type $(u, (3g)^t)$.

When $h \geq 9$, apply Construction 3.6 with a 1-cyclic 3-GDD of type $(h/3)^3$ from Corollary 2.6 to obtain an $(h, gt, 1)$ -cyclic 3-HGDD of type $(u, (gh)^t)$. \square

Lemma 3.12 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a $(3, 3, 1)$ -cyclic 3-HGDD of type $(u, 3^3)$.*

Proof When $u = 8$, let $X = I \times Z_3 \times Z_3$, $\mathcal{G} = \{\{i\} \times Z_3 \times Z_3 : i \in I\}$, $\mathcal{H} = \{I \times Z_3 \times \{j\} : j \in Z_3\}$, where $I = Z_7 \cup \{\infty\}$. Developing the following 8 initial base blocks by $(+i, +j, -) \pmod{(7, 3, -)}$ yields all 168 base blocks of the required design, where $(i, j) \in Z_7 \times Z_3$ and $\infty + 1 = \infty$.

$$\begin{aligned} &\{(0, 0, 0), (1, 1, 2), (3, 0, 1)\}, \{(0, 0, 0), (1, 0, 2), (\infty, 0, 1)\}, \{(0, 0, 0), (1, 2, 2), (3, 2, 1)\}, \\ &\{(0, 0, 0), (1, 0, 1), (3, 0, 2)\}, \{(0, 0, 0), (2, 1, 2), (\infty, 2, 1)\}, \{(0, 0, 0), (1, 1, 1), (3, 2, 2)\}, \\ &\{(0, 0, 0), (1, 2, 1), (3, 1, 2)\}, \{(0, 0, 0), (3, 1, 1), (\infty, 2, 2)\}. \end{aligned}$$

When $u \geq 14$, start with a $\text{PBD}(u, \{3, 4, 5^*\})$ from Lemma 2.3. Applying Construction 3.7 with a $(3, 3, 1)$ -cyclic 3-HGDD of type $(k, 3^3)$ for each $k \in \{3, 4, 5\}$, which exists by Example 3.2 and Lemma 3.9, we then obtain the required designs. \square

Lemma 3.13 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a $(g, t, 1)$ -cyclic 3-HGDD of type (u, g^t) for any $g \equiv t \equiv 3 \pmod{6}$.*

Proof When $t = 3$, by Lemma 3.12, there exists a $(3, 3, 1)$ -cyclic 3-HGDD of type $(u, 3^3)$. Inflating it by a 1-cyclic 3-GDD of type $(g/3)^3$ from Corollary 2.6, we then obtain a $(g, 3, 1)$ -cyclic 3-HGDD of type (u, g^3) by Construction 3.6.

When $t \geq 9$, by Lemma 3.10, there exists a $(g, t, 1)$ -cyclic 3-HGDD of type $(u, (3g)^{(t/3)})$. Filling in the holes with above $(g, 3, 1)$ -cyclic 3-HGDD of type (u, g^3) , we then obtain the required designs by Construction 3.8. \square

Dai, *et al.* [8] discussed the existence of a $(1, g, t)$ -cyclic 3-HGDDs of type (u, g^t) , which is called a g -cyclic 3-HGDDs of type (u, g^t) . We quote the result as follows.

Theorem 3.14 [8] *A $(1, g, t)$ -cyclic 3-HGDD of type (u, g^t) exists if and only if $u, t \geq 3$, $(t-1)(u-1)g \equiv 0 \pmod{2}$ and $t(t-1)u(u-1)g \equiv 0 \pmod{6}$, except for $u = t = 3$ and $g \equiv 0 \pmod{2}$.*

In fact, when $g \neq w$, we also need the k -HGDDs of type $(u, g^t w^1)$ admitting a permutation. But the general form of the permutation will be very complex. We here only permit the k -HGDDs having a simple permutation.

Let h be a common divisor of g and w , say $g = g'h$ and $w = w'h$. Suppose $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is a k -HGDD of type $(u, g^t w^1)$. If there is a permutation π on X which is the product of $u(g't + w')$ disjoint h -cycles, fixes every group and every hole, leaves \mathcal{B} invariant, then this design is said to be h -cyclic. Similarly, \mathcal{B} can be partitioned into equivalence classes under the action of π . A set of base blocks is a set of representatives for these equivalence classes.

The following result will be used in the next section to construct another auxiliary design.

Lemma 3.15 *Let $u \equiv 2 \pmod{6}$, $u \geq 8$ and $h \equiv 1 \pmod{2}$. Then there exists an h -cyclic 3-HGDD of type $(u, (3h)^{2t}(sh)^1)$ for any $t \geq 2$ and $s \in \{1, 5\}$.*

Proof Start with a 3-HGDD of type $(u, 3^{2t}s^1)$, which exists by Theorem 3.1. Inflate it by a 3-SCGDD of type h^3 from Corollary 2.6 to obtain an h -cyclic 3-HGDD of type $(u, (3h)^{2t}(sh)^1)$. \square

We next introduce a new auxiliary design, which can be considered as a generalization of a k -HGDD.

An *incomplete* HGDD (k -IHGDD) of type (u, e, g^t) is defined to be a quintuple $(X, Y, \mathcal{G}, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

- (1) X is an (ugt) -set (of *points*);
- (2) \mathcal{G} is a partition of X into u subsets (called *groups*) of points gt each;
- (3) \mathcal{H} is another partition of X into t subsets (called *holes*) of points ug each such that $|G \cap H| = g$ for any $G \in \mathcal{G}$, $H \in \mathcal{H}$;
- (4) Y is an union of e groups of \mathcal{G} ;
- (5) \mathcal{B} is a collection of k -subsets of X (called *blocks*) such that any pair of X from two distinct groups with the exception of those in which both lie in Y appears in a hole or exactly in one block but not both, and no other pairs of X occur in any block.

We also need the k -IHGDDs admitting a special permutation.

A k -IHGDD of type $(u, e, (gh)^{mt})$, $(X, Y, \mathcal{G}, \mathcal{H}, \mathcal{B})$, is said to be (h, gt, m) -cyclic, if there is a permutation π on X which is the product of uhm disjoint gt -cycles, fixes every group, leaves \mathcal{B} invariant, and partitions the mt elements of \mathcal{H} into m equivalence classes. A $(1, gt, 1)$ -cyclic k -IHGDD of type (u, e, g^t) is also referred to as a *semi-cyclic* k -IHGDD of type (u, e, g^t) and denoted by k -SCIHGDD of type (u, e, g^t) .

Without loss of generality, we always identify $X = I_u \times I_h \times I_m \times Z_{gt}$, $Y = I_e \times I_h \times I_m \times Z_{gt}$, $\mathcal{G} = \{\{i\} \times I_h \times I_m \times Z_{gt} : i \in I_u\}$ and $\mathcal{H} = \{I_u \times I_h \times \{i\} \times \{0 + j, t + j, \dots, (g-1)t + j\} : (i, j) \in I_m \times Z_t\}$. In this case, the permutation can be taken as $(x, y, z, w) \mapsto (x, y, z, w + 1) \bmod (-, -, -, gt)$. Clearly, under the action of π , \mathcal{B} can be partitioned into equivalence classes called the *block orbits*. A set of base blocks is a set of representatives for these block orbits of \mathcal{B} .

Example 3.16 *There exists a 3-SCIHGDD of type $(8, 2, 1^3)$.*

Proof Let $X = I_8 \times Z_3$, $\mathcal{G} = \{\{i\} \times Z_3 : i \in I_8\}$, $Y = \{0, 4\} \times Z_3$, and $\mathcal{H} = \{I_8 \times \{j\} : j \in Z_3\}$. Only base blocks are listed below.

$$\begin{aligned} &\{(0, 0), (1, 1), (2, 2)\}, \{(0, 0), (5, 2), (7, 1)\}, \{(0, 0), (2, 1), (3, 2)\}, \\ &\{(0, 0), (5, 1), (6, 2)\}, \{(0, 0), (1, 2), (3, 1)\}, \{(0, 0), (6, 1), (7, 2)\}, \\ &\{(1, 1), (5, 2), (4, 0)\}, \{(1, 1), (6, 0), (3, 2)\}, \{(1, 1), (6, 2), (5, 0)\}, \\ &\{(1, 1), (7, 0), (4, 2)\}, \{(1, 1), (2, 0), (7, 2)\}, \{(2, 2), (4, 1), (6, 0)\}, \\ &\{(2, 2), (4, 0), (5, 1)\}, \{(2, 2), (7, 0), (6, 1)\}, \{(2, 2), (3, 1), (5, 0)\}, \\ &\{(3, 0), (4, 1), (6, 2)\}, \{(3, 0), (7, 1), (4, 2)\}, \{(3, 0), (5, 1), (7, 2)\}. \end{aligned}$$

□

Lemma 3.17 [10] *Let $p \geq 5$ be a prime. Then there exists an element $x \in Z_p \setminus \{0, \pm 1\}$ such that x and $x + 1$ are both nonsquares and $x - 1$ is a square.*

Lemma 3.18 *There exists a 3-SCIHGDD of type $(8, 2, 1^p)$ for any odd prime p .*

Proof The case of $p = 3$ can be found in Example 3.16. We construct a 3-SCIHGDD of type $(8, 2, 1^p)$ for each prime $p \geq 5$ below. Let $X = I \times Z_p$, $\mathcal{G} = \{\{i\} \times Z_p : i \in I\}$, $Y = \{a, b\} \times Z_p$, and $\mathcal{H} = \{I \times \{j\} : j \in Z_p\}$, where $I = Z_6 \cup \{a, b\}$. By Lemma 3.17, we can take $x \in Z_p \setminus \{0, \pm 1\}$ such that x and $x + 1$ are both nonsquares, while $x - 1$ is a square. Only initial base blocks are listed below.

$p \equiv 1 \pmod{4}$:

$\{(1+2i, 0), (3+2i, x), (a, x+1)\}, \{(2i, 0), (1+2i, 1), (3+2i, x)\},$
 $\{(2i, 0), (3+2i, 1), (b, x+1)\}, \{(2i, 0), (2+2i, 1), (a, x)\},$
 $\{(1+2i, 0), (2+2i, x), (b, 1)\}, \{(2i, 0), (1+2i, x), (2+2i, x+1)\}.$

$p \equiv 3 \pmod{4}$:

$\{(2i, 0), (1+2i, x+1), (3+2i, 1)\}, \{(2i, 0), (2+2i, 1), (a, x)\},$
 $\{(2i, 0), (1+2i, 1), (2+2i, x)\}, \{(2i, 0), (3+2i, x), (b, x-1)\},$
 $\{(1+2i, 0), (3+2i, x), (a, x+1)\}, \{(1+2i, 0), (2+2i, x), (b, 1)\}.$

Here, $0 \leq i \leq 2$. All $9(p-1)$ base blocks are generated by multiplying above 18 initial base blocks by ω^{2r} , where ω is a primitive element of Z_p , $0 \leq r \leq (p-3)/2$. \square

We give two constructions on k -SCIHGDDs, which can be considered as the generalizations of Constructions 3.1 and 3.4 of [10] respectively.

Construction 3.19 *If there exist a k -SCIHGDD of type (u, e, g^t) and a k -SCGDD of type h^k , then there exists a k -SCIHGDD of type $(u, e, (gh)^t)$.*

Construction 3.20 *If there exist a k -SCIHGDD of type $(u, e, (gh)^t)$ and a k -SCIHGDD of type (u, e, g^h) , then there exists a k -SCIHGDD of type (u, e, g^{ht}) .*

Applying these constructions, we give an infinite class of 3-SCIHGDDs.

Lemma 3.21 *There exists a 3-SCIHGDD of type $(8, 2, 1^t)$ for any odd integers $t \geq 3$.*

Proof Let $t = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$, where $p_i \geq 3$ is a prime for each $1 \leq i \leq s$. Start from a 3-SCIHGDD of type $(8, 2, 1^{p_1})$, which exists by Lemma 3.18. Inflate it by a 3-SCGDD of type q^3 from Corollary 2.6, $q \in \{p_1, p_2, \dots, p_s\}$. By Construction 3.19, we obtain a 3-SCIHGDD of type $(8, 2, q^{p_1})$. Fill in the holes with a 3-SCIHGDD of type $(8, 2, 1^q)$, which exists by Lemma 3.18. Apply Construction 3.20 to obtain a 3-SCIHGDD of type $(8, 2, 1^{p_1 q})$. Repeating this process will produce the required designs for any odd integers $t \geq 3$. \square

Lemma 3.22 *There exists a $(1, t, h)$ -cyclic 3-IHGDD of type $(8, 2, 1^{th})$ for any odd positive integers h and $t \geq 3$.*

Proof By Lemma 3.21, there exists a 3-SCIHGDD of type $(8, 2, 1^{th})$, which is also a $(1, t, h)$ -cyclic 3-IHGDD of type $(8, 2, 1^{th})$. \square

4 Incomplete group divisible packings

Incomplete group divisible designs (IGDDs) are important auxiliary designs in the constructions of GDDs. In this section, we generalize this concept to incomplete group divisible packings (IGDPs), which will be used to construct w -cyclic GDPs.

An *incomplete group divisible packing* (K -IGDP) is a quadruple $(X, Y, \mathcal{G}, \mathcal{B})$ where X is a set (of *points*), Y is a subset (called a *hole*) of X , \mathcal{G} is a partition of X into subsets (called *groups*), and \mathcal{B} is a collection of subsets (called *blocks*) of X each of size from K such that

- (1) each block intersects each group in at most one point;
- (2) no pair of distinct points of Y occurs in any block;
- (3) every pair of points from distinct groups with the exception of those in which both lie in Y occurs in at most one block of \mathcal{B} .

If $K = \{k\}$, we omit the braces to simply write k for K . If every admissible pair appears in exactly one block, then this IGDP is called an *incomplete group divisible design*, or a K -IGDD for short.

We introduce two types of IGDPs here.

- A K -IGDP of type $(g, t)^u$ is an IGDP in which each block has size from K and there are u groups of size g , each of which intersects the hole in t points.
- A K -IGDP of type $g^{(u, t)}$ is an IGDP in which each block has size from K and there are u groups of size g , wherein t groups constitute the hole.

In next subsections, we study the constructions of these two types of K -IGDPs with special permutations and give some existence results for later use.

4.1 h -cyclic K -IGDPs of type $(gh, th)^u$

We need the K -IGDPs of type $(gh, th)^u$ admitting a special permutation.

A K -IGDP of type $(gh, th)^u$ $(X, Y, \mathcal{G}, \mathcal{B})$ is said to be *h -cyclic* if there is a permutation on X which is the product of ug disjoint h -cycles, fixes every group, and leaves Y, \mathcal{B} invariant. Without loss of generality, we can always identify $X = I_u \times I_g \times Z_h$, $\mathcal{G} = \{\{i\} \times I_g \times Z_h : i \in I_u\}$ and $Y = I_u \times I_t \times Z_h$. In this case, the permutation can be taken as $(i, j, x) \mapsto (i, j, x + 1) \bmod (-, -, h)$. \mathcal{B} can be partitioned into some block orbits under the permutation. A set of base blocks is a set of representatives for these block orbits of \mathcal{B} .

We exhibit our second “Filling Construction” via h -cyclic k -IGDPs of type $(gh, th)^u$.

Construction 4.1 [24]/(Filling Construction-II) *Suppose that the following designs exist:*

- (1) an h -cyclic k -IGDP of type $(gh, th)^u$ with f base blocks;
- (2) an h -cyclic k -GDP of type $(th)^u$ with b base blocks.

Then, there exists an h -cyclic k -GDP of type $(gh)^u$ with $b + f$ base blocks.

For applying Construction 4.1, we need to construct more h -cyclic K -IGDPs of type $(gh, th)^u$. At the beginning, we give some constructive methods. Construction 4.2 can be found in [24]. Constructions 4.3 and 4.4 are simple generalizations of the corresponding constructions in [24].

Construction 4.2 (Weighting Construction-I) *Suppose there exist a K -GDD of type m^u and an h -cyclic l -IGDD of type $(gh, th)^k$ for each $k \in K$. Then there exists an h -cyclic l -IGDD of type $(mgh, mth)^u$.*

Construction 4.3 (Inflation-II) Suppose that there exists an h -cyclic K -IGDD of type $(gh, th)^u$. Let \mathcal{F} be the base block set of this design. Suppose \mathcal{F} contains r_i base blocks of size k_i , $1 \leq i \leq s$. If there exists a w -cyclic k -GDP of type $(vw)^{k_i}$ with b_i base blocks for each $1 \leq i \leq s$, then there exists an hw -cyclic k -IGDP of type $(ghvw, thvw)^u$ with $\sum_{i=1}^s r_i b_i$ base blocks.

Remark 1: For each base block of size k_i in \mathcal{F} , if the input design is a w -cyclic k -GDD of type $(vw)^{k_i}$ in Construction 4.3, then the output design is an hw -cyclic k -IGDD of type $(ghvw, thvw)^u$.

Construction 4.4 Suppose there exists a (g, h, t) -cyclic k -HGDD of type $(u, (gh)^t)$ with b base blocks and an h -cyclic k -IGDP of type $(gh + eh, eh)^u$ with f base blocks. Then there exists an h -cyclic k -IGDP of type $(gth + eh, eh)^u$ with $b + tf$ base blocks.

Remark 2: If the input design in Construction 4.4 is an h -cyclic k -IGDD of type $(gh + eh, eh)^u$, then the output design is an h -cyclic k -IGDD of type $(gth + eh, eh)^u$.

When w is even, Construction 4.3 some times cannot work. Therefore, we introduce a special kind of k -IGDD called an h -perfect k -IGDD as follows.

Suppose that $\mathcal{F} = \{B_i : 1 \leq i \leq r\}$ is a set of base blocks of an hw -cyclic k -IGDD of type $(ghw, thw)^u$ on $I_u \times I_g \times Z_{hw}$. Without loss of generality, let

$$B_i = \{(a_{i1}, b_{i1}, 0), (a_{i2}, b_{i2}, c_{i2}), \dots, (a_{ik}, b_{ik}, c_{ik})\}$$

for each $1 \leq i \leq r$. Define

$$ele(\mathcal{F}) = \cup_{i=1}^r \{c_{i2}, c_{i3}, \dots, c_{ik}\}.$$

The hw -cyclic k -IGDD of type $(ghw, thw)^u$ is said to be h -perfect, denoted by h -perfect hw -cyclic k -IGDD of type $(ghw, thw)^u$, if

$$ele(\mathcal{F}) \subset \{x + yw : 0 \leq x \leq \lfloor w/2 \rfloor, 0 \leq y \leq h - 1\}.$$

When $h = 1$, a 1-perfect k -IGDD is simply called a perfect k -IGDD. Note that any hw -cyclic k -IGDD of type $(ghw, thw)^u$ has the h -perfect property when w is 2.

Construction 4.5 Suppose that the following designs exist:

- (1) a perfect w -cyclic k -IGDD of type $(gw, tw)^u$;
- (2) an h -perfect hw -cyclic k -IGDD of type $(ghw, thw)^u$;
- (3) a k -SCHGDD of type (k, h^m) .

Then, there exists an hm -perfect hmw -cyclic k -IGDD of type $(ghmw, thmw)^u$.

Proof Suppose that $\mathcal{F} = \{(a_{i1}, b_{i1}, 0), (a_{i2}, b_{i2}, c_{i2}), \dots, (a_{ik}, b_{ik}, c_{ik})\} : 1 \leq i \leq r\}$ be a set of base blocks of a perfect w -cyclic k -IGDD of type $(gw, tw)^u$, where $0 \leq c_{ij} \leq \lfloor w/2 \rfloor$ for each $1 \leq i \leq r$ and $2 \leq j \leq k$.

Let $\mathcal{A} = \{(q_{i1}, f_{i1}, 0), (q_{i2}, f_{i2}, x_{i2} + y_{i2}w), \dots, (q_{ik}, f_{ik}, x_{ik} + y_{ik}w)\} : 1 \leq i \leq s\}$ be a set of base blocks of an h -perfect hw -cyclic k -IGDD of type $(ghw, thw)^u$, where $0 \leq x_{ij} \leq \lfloor w/2 \rfloor$ and $0 \leq y_{ij} \leq h - 1$ for each $1 \leq i \leq s$ and $2 \leq j \leq k$.

Let $\mathcal{E} = \{(1, d_{1j}), (2, d_{2j}), \dots, (k, d_{kj}) : 1 \leq j \leq h(m-1)\}$ be the base block set of a k -SCHGDD of type (k, h^m) . Then $\{d_{\alpha j} - d_{\beta j} : 1 \leq j \leq h(m-1)\} = Z_{hm} \setminus \{0, m, \dots, (h-1)m\}$ for any $1 \leq \alpha \neq \beta \leq k$.

Now we construct the desired hm -perfect hmw -cyclic k -IGDD of type $(ghmw, thmw)^u$ on $I_u \times I_g \times Z_{hmw}$ whose base blocks consists of the following two parts:

(1) For each $B_i = \{(a_{i1}, b_{i1}, 0), (a_{i2}, b_{i2}, c_{i2}), \dots, (a_{ik}, b_{ik}, c_{ik})\} \in \mathcal{F}$, construct a family

$$\mathcal{C}_{B_i}^j = \{(a_{i1}, b_{i1}, 0), (a_{i2}, b_{i2}, c_{i2} + (d_{2j} - d_{1j})w), \dots, (a_{ik}, b_{ik}, c_{ik} + (d_{kj} - d_{1j})w)\},$$

where $1 \leq j \leq h(m-1)$ and the third coordinates are reduced modulo hmw .

(2) For each $A_i = \{(q_{i1}, f_{i1}, 0), (q_{i2}, f_{i2}, x_{i2} + y_{i2}w), \dots, (q_{ik}, f_{ik}, x_{ik} + y_{ik}w)\} \in \mathcal{A}$, let

$$A'_i = \{(q_{i1}, f_{i1}, 0), (q_{i2}, f_{i2}, x_{i2} + y_{i2}mw), \dots, (q_{ik}, f_{ik}, x_{ik} + y_{ik}mw)\}.$$

Let $\mathcal{C}_1 = \cup_{i=1}^r \cup_{j=1}^{h(m-1)} \mathcal{C}_{B_i}^j$, $\mathcal{C}_2 = \cup_{i=1}^s A'_i$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. It is readily checked that \mathcal{C} forms a family of base blocks of the required hm -perfect hmw -cyclic k -IGDD of type $(ghmw, thmw)^u$. \square

Applying these constructions, we give some results on h -cyclic 3-IGDDs of type $(gh, th)^u$ for later use. The results in Lemma 4.6 are taken from Lemmas 4.12 – 4.15 of [24].

Lemma 4.6 [24] *There exists a w -cyclic 3-IGDD of type $(vw, w)^u$ if u, v, w satisfy one of the following conditions.*

- (1) $(u, v, w) = (5, 2, 4)$;
- (2) $u \in \{3, 4\}$, $w = 2$ and $v \equiv 1 \pmod{2}$;
- (3) $u = 5$, either $w = 2$ and $v \equiv 1, 2 \pmod{3}$, or $w = 6$ and $v \equiv 1 \pmod{2}$.

Lemma 4.7 *There exists a w -cyclic 3-IGDD of type $(vw, w)^{11}$ for any $v \equiv 1, 5 \pmod{6}$ and $w \equiv 10 \pmod{12}$.*

Proof Start with a PBD(11, $\{3, 5^*\}$), which is also a 3-GDD of type $1^6 5^1$ from Lemma 2.1. Apply Construction 4.2 with 2-cyclic 3-IGDDs of type $(2v, 2)^k$ for $k \in \{3, 5\}$ from Lemma 4.6 to obtain a 2-cyclic 3-IGDD of type $(2v, 2)^{11}$. Inflate it by a 3-SCGDD of type $(w/2)^3$ from Corollary 2.6. By Construction 4.3, we obtain a w -cyclic 3-IGDD of type $(vw, w)^{11}$. \square

Lemma 4.8 *Let $w \equiv 0 \pmod{2}$. Then there exists a w -cyclic 3-IGDD of type $(2w, w)^8$.*

Proof Let $w = 2^n w'$, where $n \geq 1$ and $w' \equiv 1 \pmod{2}$. By Lemma A.1 in Supporting Information, there exist a perfect 2-cyclic 3-IGDD of type $(4, 2)^8$ and a 2-perfect 4-cyclic 3-IGDD of type $(8, 4)^8$. Applying Construction 4.5 with a 3-SCHGDD of type $(3, 2^{2^{n-2}})$ from Theorem 3.5 for any $n \geq 4$, we obtain a 2^{n-1} -perfect 2^n -cyclic 3-IGDD of type $(2^{n+1}, 2^n)^8$.

Combining the 8-cyclic 3-IGDD of type $(16, 8)^8$ from Lemma A.1 in Supporting Information, we have a 2^n -cyclic 3-IGDD of type $(2^{n+1}, 2^n)^8$ for all $n \geq 1$. Inflate it by a 3-SCGDD of type w'^3 from Corollary 2.6, we obtain the required w -cyclic 3-IGDD of type $(2w, w)^8$. \square

Lemma 4.9 *Let $v \equiv 1, 2 \pmod{3}$ and $v \geq 2$. Then there exists a w -cyclic 3-IGDD of type $(vw, w)^8$ for any $w \equiv 0 \pmod{2}$.*

Proof When $v = 2$, a w -cyclic 3-IGDD of type $(2w, w)^8$ exists by Lemma 4.8. When $v \geq 4$, start with a $(1, w, v - 1)$ -cyclic 3-HGDD of type $(8, w^{v-1})$ from Theorem 3.14. Apply Construction 4.4 with a w -cyclic 3-IGDD of type $(2w, w)^8$ to obtain the required w -cyclic 3-IGDD of type $(vw, w)^8$. \square

Lemma 4.10 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a 4-cyclic 3-IGDD of type $(4v, 4)^u$ for any $v \equiv 1, 2 \pmod{3}$.*

Proof (1) $v = 2$. For $u = 5$, Lemma 4.6 provides the required design. For $u \in \{4, 6, 8, 14\}$, Lemmas A.1 and A.2 in Supporting Information provide the required designs. For $u \geq 20$, start with a PBD($u, \{4, 5, 6\}$), which exists by Lemma 2.2. Apply Construction 4.2 with a 4-cyclic 3-IGDD of type $(8, 4)^k$ for $k \in \{4, 5, 6\}$ to obtain the required design.

(2) $v \geq 4$. By Theorem 3.14, there exists a $(1, 4, v - 1)$ -cyclic 3-HGDD of type $(u, 4^{v-1})$. Apply Construction 4.4 with a 4-cyclic 3-IGDD of type $(8, 4)^u$ to obtain the required design. \square

Lemma 4.11 *Suppose $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a 2-cyclic 3-IGDD of type $(2v, 2)^u$ for any $v \equiv 1, 5 \pmod{6}$.*

Proof Start with a PBD($u, \{3, 4, 5^*\}$) from Lemma 2.3. Apply Construction 4.2 with 2-cyclic 3-IGDDs of type $(2v, 2)^k$ for $k \in \{3, 4, 5\}$, which exists by Lemma 4.6, to obtain the required design. \square

Lemma 4.12 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a w -cyclic 3-IGDD of type $(vw, w)^u$ for any $w \equiv 10 \pmod{12}$ and $v \equiv 1, 5 \pmod{6}$.*

Proof By Lemma 4.11, a 2-cyclic 3-IGDD of type $(2v, 2)^u$ exists. Inflate it by a 3-SCGDD of type $(w/2)^3$ from Corollary 2.6. Then we obtain a w -cyclic 3-IGDD of type $(vw, w)^u$ by Construction 4.3. \square

Lemma 4.13 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a 6-cyclic 3-IGDD of type $(6v, 6)^u$ for any $v \equiv 1 \pmod{2}$.*

Proof For each $k \in \{3, 4\}$, take a 2-cyclic 3-IGDD of type $(2v, 2)^k$ from Lemma 4.6. Inflate it by a 3-SCGDD of type 3^3 from Corollary 2.6 to obtain a 6-cyclic 3-IGDD of type $(6v, 6)^k$. When $k = 5$, a 6-cyclic 3-IGDD of type $(6v, 6)^5$ can be found in Lemma 4.6. Apply Construction 4.2 with a PBD($u, \{3, 4, 5^*\}$) to obtain the required designs. \square

Corollary 4.14 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a 2-cyclic 3-IGDD of type $(6v, 6)^u$ for any $v \equiv 1 \pmod{2}$.*

Proof By Lemma 4.13, there exists a 6-cyclic 3-IGDD of type $(6v, 6)^u$, which is also a 2-cyclic 3-IGDD of type $(6v, 6)^u$. \square

At the end of this subsection, we present two existence results on h -cyclic 3-IGDPs of type $(gh, th)^u$.

Lemma 4.15 *Suppose that $u \in \{8, 14\}$ and $s \in \{1, 5\}$. Let $v = 6t + s$ and $t \geq 1$. Then there exists a 3-cyclic 3-IGDP of type $(3v, 3s)^u$ with $u(v-s)(3(u-1)(v+s)-1)/6$ base blocks.*

Proof When $t = 1$, by Lemma A.3 in Supporting Information, there exists a $\{3, u-2\}$ -IGDD of type $(v, s)^u$, in which the blocks of size $u-2$ form a partition of the points outside the hole. Give weight 3 to each point and input 3-SCGDDs of type 3^3 or 3-SCGDPs of type 3^{u-2} with $J^*((u-2) \times 1 \times 3)$ base blocks, which can be found in Corollary 2.6. By Construction 4.3, we obtain a 3-cyclic 3-IGDP of type $(3v, 3s)^u$, which has $u(3(u-1)(v+s)-1)$ base blocks.

When $t \geq 2$, start from a 3-cyclic 3-HGDD of type $(u, 9^{2t}(3s)^1)$, which has $3u(u-1)t(6t+2s-3)$ base blocks and exists by Lemma 3.15. Fill in the $2t$ holes of size $9u$ using 3-cyclic 3-GDPs of type 9^u from Lemma 5.8, each of which has $J^*(u \times 3 \times 3, 3, 1)$ base blocks, to obtain a 3-cyclic 3-IGDP of type $(3v, 3s)^u$ with $3u(u-1)t(6t+2s-3) + ut(9u-10) = u(v-s)(3(u-1)(v+s)-1)/6$ base blocks. Note that $J^*(u \times 3 \times 3, 3, 1) = u(9u-10)/2$. \square

Lemma 4.16 *Let $u \in \{8, 14\}$. Then there exists a w -cyclic 3-IGDP of type $(vw, w)^u$ with $u(v-1)((u-1)(v+1)w-1)/6$ base blocks for any $v \equiv 1 \pmod{6}$, $v \geq 7$ and $w \equiv 5 \pmod{6}$.*

Proof The construction is similar to that of Lemma 4.15. When $v = 7$, start also with the $\{3, u-2\}$ -IGDD of type $(7, 1)^u$ from Lemma A.3 in Supporting Information. Give weight w to each point and input 3-SCGDDs of type w^3 or 3-SCGDPs of type w^{u-2} with $J^*((u-2) \times 1 \times w)$ base blocks, which exist by Corollary 2.6.

When $v \geq 13$, start with a w -cyclic 3-HGDD of type $(u, (3w)^{(v-1)/3}w^1)$ from Lemma 3.15. For each hole of size $3wu$, construct a w -cyclic 3-GDP of type $(3w)^u$ with $J^*(u \times 3 \times w, 3, 1)$ base blocks, which exists by Lemma 5.16. It is readily checked that we obtain a w -cyclic 3-IGDP of type $(vw, w)^u$ with $u(v-1)((u-1)(v+1)w-1)/6$ base blocks. \square

Remark 3: The results of Lemmas 5.8 and 5.16 are used in Lemmas 4.15 and 4.16, respectively. Note that the constructions of Lemmas 5.8 and 5.16 only need the conclusions of Section III.

4.2 h -cyclic K -IGDPs of type $(gh)^{(u,t)}$

We also need the K -IGDPs of type $(gh)^{(u,t)}$ admitting a special permutation.

A K -IGDP of type $(gh)^{(u,t)}$ $(X, Y, \mathcal{G}, \mathcal{B})$ is said to be h -cyclic if there is a permutation on X which is also a product of ug disjoint h -cycles, fixes every group, and leaves Y , \mathcal{B} invariant. Without loss of generality, we can always identify $X = I_u \times I_g \times Z_h$, $\mathcal{G} = \{\{i\} \times I_g \times Z_h : i \in I_u\}$ and $Y = I_t \times I_g \times Z_h$. In this case, the permutation can be taken as $(i, j, x) \mapsto (i, j, x+1) \pmod{(-, -, h)}$. Also, \mathcal{B} can be partitioned into some block orbits under the permutation. A set of base blocks is a set of representatives for these block orbits of \mathcal{B} .

We exhibit our third “Filling Construction” via h -cyclic k -IGDPs of type $(gh)^{(u,t)}$.

Construction 4.17 [24]/(Filling Construction-III) Suppose that the following designs exist:

- (1) an h -cyclic k -IGDP of type $(gh)^{(u,t)}$ with b base blocks;
- (2) an h -cyclic k -GDP of type $(gh)^t$ with f base blocks.

Then, there exists an h -cyclic k -GDP of type $(gh)^u$ with $b + f$ base blocks.

We introduce some constructive methods for h -cyclic k -IGDPs of type $(gh)^{(u,t)}$. The first three constructions are taken from [24]. The last construction is a simple generalization of Construction 4.19 of [24].

Construction 4.18 (Inflation-III) Suppose there exists an h -cyclic K -IGDD of type $(gh)^{(u,t)}$. If there exists a w -cyclic l -GDD of type $(vw)^k$ for each $k \in K$, then there exists an hw -cyclic l -IGDD of type $(ghvw)^{(u,t)}$.

Construction 4.19 (Weighting Construction-II) Suppose there exists a K -GDD of type $g_1^{u_1} \cdots g_r^{u_r}$. If there exists an h -cyclic l -GDD of type $(ht)^k$ for each $k \in K$, then there exists an h -cyclic l -GDD of type $(g_1ht)^{u_1} \cdots (g_rht)^{u_r}$.

Construction 4.20 Suppose that there exists a $PBD(u, K \cup \{k^*\})$. If there exist a g -cyclic l -IGDD of type $g^{(u,k)}$ and an l -SCHGDD of type (n, g^t) for each $n \in K$, then there exists a gt -cyclic l -IGDD of type $(gt)^{(u,k)}$.

Construction 4.21 (Filling Construction-IV) Suppose that the following designs exist:

- (1) an h -cyclic k -GDD of type $\{ght_i : i = 1, 2, \dots, r\}$ with f base blocks;
- (2) an h -cyclic k -IGDP of type $(gh)^{(t_i+t,t)}$ with b_i base blocks for each $1 \leq i \leq r-1$.

Then there exists an h -cyclic k -IGDP of type $(gh)^{(u+t, t_r+t)}$ with $f + \sum_{i=1}^{r-1} b_i$ base blocks, where $u = \sum_{i=1}^r t_i$. Furthermore, if an h -cyclic k -IGDP of type $(gh)^{(t_r+t,t)}$ with b_r base blocks exists, then an h -cyclic k -IGDP of type $(gh)^{(u+t,t)}$ with $f + \sum_{i=1}^r b_i$ base blocks exists.

Remark 4: If the input design in Construction 4.21 is an h -cyclic k -IGDD of type $(gh)^{(t_i+t,t)}$ for each $1 \leq i \leq r-1$, then the output design is an h -cyclic k -IGDD of type $(gh)^{(u+t, t_r+t)}$. If further input an h -cyclic k -IGDD of type $(gh)^{(t_r+t,t)}$, then the output is an h -cyclic k -IGDD of type $(gh)^{(u+t,t)}$.

We summarize our results on h -cyclic 3-IGDDs of type $(gh)^{(u,t)}$ as follows.

Lemma 4.22 [24] There exists a 2-cyclic 3-IGDD of type $2^{(7,3)}$.

Lemma 4.23 [28] If $(m, d) \equiv (2, 0), (2, 1), (1, 0), (3, 1) \pmod{(4, 2)}$ and $m(m-2d+1) + 2 \geq 0$, then $[d, d+3m] \setminus \{d+3m-1\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq m$ such that $a_i + b_i = c_i$.

Lemma 4.24 There exists a 2-cyclic 3-IGDD of type $2^{(u,5)}$ for any $u \equiv 8 \pmod{12}$ and $u \geq 20$.

Proof For $u \in \{20, 32, 44, 56\}$, the required 2-cyclic 3-IGDD of type $2^{(u,5)}$ can be found in Lemma B.1 in Supporting Information.

When $u \geq 68$, let $d = 8$ and $m = (u - 14)/3$. Then $m \geq 18$, $(m, d) \equiv (2, 0) \pmod{(4, 2)}$ and $m(m - 2d + 1) + 2 \geq 0$. By Lemma 4.23, we partition $[8, 8 + 3m] \setminus \{3m + 7\}$ into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq m$ such that $a_i + b_i = c_i$.

Let $X = Z_{2(u-5)} \cup (I_5 \times Z_2)$, $\mathcal{G} = \{\{i, u - 5 + i\} : 0 \leq i \leq u - 6\} \cup \{\{i \times Z_2\} : i \in I_5\}$, and $Y = I_5 \times Z_2$. Let \mathcal{F} contains following $m + 6$ base blocks:

$$\begin{aligned} &\{0, a_j, c_j\}, 1 \leq j \leq m; \\ &\{0, u - 7, (4, 0)\}, \{0, 2, 6\}; \\ &\{0, (2i + 1), (i, 0)\}, 0 \leq i \leq 3. \end{aligned}$$

Developing these base blocks by $+1 \pmod{2(u - 5)}$ yields all blocks, where $(i, x) + 1 \equiv (i, x + 1) \pmod{(-, 2)}$. It is readily checked that the designs are isomorphism to 2-cyclic 3-IGDDs of type $2^{(u,5)}$. \square

Lemma 4.25 *There exists a 2-cyclic 3-IGDD of type $2^{(u,t)}$ if u, t satisfy one of the following conditions.*

- (1) $u \equiv 2 \pmod{12}$, $u \geq 38$ and $t = 14$;
- (2) $u \equiv 11 \pmod{12}$, $u \geq 23$ and $t = 11$.

Proof (1) Start with a 2-cyclic 3-GDD of type $24^{(u-2)/12}$ from Corollary 2.6. Fill in the groups with a 2-cyclic 3-IGDD of type $2^{(14,2)}$ from Lemma B.2 in Supporting Information. By Construction 4.21, we obtain a 2-cyclic 3-IGDD of type $2^{(u,14)}$.

(2) By Lemma 2.1, there exists a 3-GDD of type $2^{(u-11)/4}4^1$. Apply Construction 4.19 with a 2-cyclic 3-GDD of type 4^3 to obtain a 2-cyclic 3-GDD of type $8^{(u-11)/4}16^1$. Filling in the groups with a 2-cyclic 3-IGDD of type $2^{(7,3)}$ from Lemma 4.22, by Construction 4.21, then we obtain a 2-cyclic 3-IGDD of type $2^{(u,11)}$. \square

Lemma 4.26 *Let $u \equiv 11 \pmod{12}$ and $u \geq 23$. Then there exists a w -cyclic 3-IGDD of type $(vw)^{(u,11)}$ for any $v \equiv 1, 5 \pmod{6}$ and $w \equiv 10 \pmod{12}$.*

Proof Start with a PBD($u, \{3, 11^*\}$), which is also a 3-GDD of type $1^{u-11}11^1$ from Lemma 2.1. Apply Construction 4.20 with a 3-SCHGDD of type $(3, 2^{w/2})$ from Theorem 3.5 and a 2-cyclic 3-IGDD of type $2^{(u,11)}$ from Lemma 4.25 to obtain a w -cyclic 3-IGDD of type $w^{(u,11)}$. Further apply Construction 4.18 with a 1-cyclic 3-GDD of type v^3 from Corollary 2.6 to obtain a w -cyclic 3-IGDD of type $(vw)^{(u,11)}$. \square

Lemma 4.27 *There exists a 4-cyclic 3-IGDD of type $4^{(u,2)}$ for any $u \equiv 2 \pmod{6}$ and $u \geq 8$.*

Proof When $u \in \{8, 14\}$, by Lemma B.3 in Supporting Information, there exists a 4-cyclic 3-IGDD of type $4^{(u,2)}$. When $u \geq 20$, start with a 4-cyclic 3-GDD of type $24^{(u-2)/6}$ from Corollary 2.6. Apply Construction 4.21 with a 4-cyclic 3-IGDD of type $4^{(8,2)}$ to obtain a 4-cyclic 3-IGDD of type $4^{(u,2)}$. \square

Lemma 4.28 *There exists a 6-cyclic 3-IGDD of type $6^{(u,6)}$ for any $u \equiv 2 \pmod{12}$ and $u \geq 14$.*

Proof Start with a 6-cyclic 3-GDD of type $24^{(u-2)/4}$ from Corollary 2.6. Take a 6-cyclic 3-IGDD of type $6^{(6,2)}$, which can be found in Lemma B.4 in Supporting Information, to fill in the groups. We then obtain the required design by Construction 4.21. \square

Corollary 4.29 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a 2-cyclic 3-IGDD of type $6^{(u,6)}$.*

Proof By Lemma 4.28, there exists a 6-cyclic 3-IGDD of type $6^{(u,6)}$, which is also a 2-cyclic 3-IGDD of type $6^{(u,6)}$. \square

Lemma 4.30 *A w -cyclic 3-IGDD of type $(vw)^{(u,5)}$ exists if u, v, w satisfy one of following conditions.*

- (1) $u \equiv 2 \pmod{12}$, $u \geq 14$, $v \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{2}$;
- (2) $u \equiv 8 \pmod{12}$, $u \geq 20$, $v \geq 1$, $w \equiv 0 \pmod{2}$ and $w \notin \{4, 6\}$.

Proof (1) Start with a PBD($u, \{3, 4, 5^*\}$), which is also a 1-cyclic $\{3, 4\}$ -IGDD of type $1^{(u,5)}$. Give weight vw to each point. Input a w -cyclic 3-GDD of type $(vw)^k$ for each $k \in \{3, 4\}$ from Corollary 2.6. By Construction 4.18, we then obtain a w -cyclic 3-IGDD of type $(vw)^{(u,5)}$.

(2) When $w = 2$, by Lemma 4.24, there exists a 2-cyclic 3-IGDD of type $2^{(u,5)}$. Inflate it by a 1-cyclic 3-GDD of type v^3 from Corollary 2.6 to obtain the required design.

When $w \geq 8$, similar to the proof of Lemma 4.26, we start with a PBD($u, \{3, 4, 5^*\}$). The required 2-cyclic 3-IGDD of type $2^{(u,5)}$ and 3-SCHGDDs of type $(k, 2^{w/2})$ for $k \in \{3, 4\}$ can be found in Lemma 4.24 and Theorem 3.5, respectively. \square

We give an existence result on h -cyclic 3-IGDPs of type $(gh)^{(u,t)}$ at the end of this subsection.

Lemma 4.31 *Let $u \equiv 2 \pmod{6}$ and $u \geq 20$. Then there exists a w -cyclic 3-IGDP of type $(vw)^{(u,8)}$ with $v(u-8)((u+7)vw-1)/6$ base blocks for any odd integers v and $w \geq 3$.*

Proof By Lemma 3.22, there exists a $(1, w, v)$ -cyclic 3-IHGDD of type $(8, 2, 1^{vw})$, which has $9v(vw-1)$ base blocks. Take a 3-GDD of type 2^4 from Lemma 2.1, which has 8 blocks and can be considered as a 1-cyclic 3-IGDP of type $1^{(8,2)}$. Fill in the holes of the IHGDD to obtain a w -cyclic 3-IGDP of type $(vw)^{(8,2)}$. Clearly, it has $9v(vw-1)+8v = v(9vw-1)$ base blocks.

When $u \geq 20$, by Corollary 2.6, there exists a w -cyclic 3-GDD of type $(6vw)^{(u-2)/6}$, which has $(u-2)(u-8)v^2w/6$ base blocks. Filling in $(u-8)/6$ groups of this GDD by above w -cyclic 3-IGDP of type $(vw)^{(8,2)}$ with $v(9vw-1)$ base blocks, then we obtain a w -cyclic 3-IGDP of type $(vw)^{(u,8)}$ with $v(u-8)((u+7)vw-1)/6$ base blocks by Construction 4.21. \square

5 Main result

In this section, applying the filling constructions established in above sections, we obtain the main result of this paper.

5.1 The possible exceptions in Corollary 2.6

We first introduce a new auxiliary design, which plays an important role in the constructions of some optimal w -cyclic 3-GDPs.

A 3-SCGDP* of type $2^{(u,t)}$ is a 3-SCGDP of type 2^u on $I_u \times Z_2$ with group set $\{\{i\} \times Z_2 : i \in I_u\}$ in which, for any $a, b \in I_t$ and $x \in Z_2$, all pairs $\{(a, x), (b, x + 1)\}$ are not covered by any block.

Example 5.1 *There exists a 3-SCGDP* of type $2^{(11,5)}$ with 32 base blocks.*

Proof Let $X = I_{11} \times Z_2$, $\mathcal{G} = \{\{i\} \times Z_2 : i \in I_{11}\}$. Only base blocks are listed below.

$\{(0, 0), (8, 1), (10, 1)\}, \{(0, 0), (3, 0), (4, 0)\}, \{(0, 0), (5, 0), (6, 0)\}, \{(0, 0), (7, 0), (8, 0)\},$
 $\{(0, 0), (9, 0), (10, 0)\}, \{(0, 0), (5, 1), (7, 1)\}, \{(0, 0), (6, 1), (9, 1)\}, \{(0, 0), (1, 0), (2, 0)\},$
 $\{(1, 0), (7, 1), (10, 1)\}, \{(1, 0), (4, 0), (6, 0)\}, \{(1, 0), (7, 0), (9, 0)\}, \{(1, 0), (8, 0), (5, 1)\},$
 $\{(1, 0), (10, 0), (6, 1)\}, \{(1, 0), (3, 0), (5, 0)\}, \{(1, 0), (8, 1), (9, 1)\}, \{(2, 0), (3, 0), (7, 0)\},$
 $\{(2, 0), (10, 0), (9, 1)\}, \{(2, 0), (5, 0), (9, 0)\}, \{(2, 0), (6, 0), (7, 1)\}, \{(2, 0), (4, 0), (8, 0)\},$
 $\{(2, 0), (6, 1), (10, 1)\}, \{(2, 0), (5, 1), (8, 1)\}, \{(3, 0), (6, 0), (8, 1)\}, \{(3, 0), (8, 0), (9, 1)\},$
 $\{(4, 0), (10, 0), (8, 1)\}, \{(4, 0), (9, 0), (5, 1)\}, \{(4, 0), (7, 0), (9, 1)\}, \{(4, 0), (6, 1), (7, 1)\},$
 $\{(4, 0), (5, 0), (10, 1)\}, \{(3, 0), (10, 0), (7, 1)\}, \{(3, 0), (5, 1), (10, 1)\}, \{(3, 0), (9, 0), (6, 1)\}.$ \square

Lemma 5.2 *Let $w \equiv 10 \pmod{12}$. Then there exists a 3-SCGDP of type w^{11} with $J^*(11 \times 1 \times w, 3, 1)$ base blocks.*

Proof Start with a PBD(11, {3, 5*}) (I_{11}, \mathcal{B}) , which is also a 3-GDD of type $1^6 5^1$ from Lemma 2.1. Without loss of generality, let I_5 be the block of size 5. For each $B \in \mathcal{B}$ of size 3, construct a 3-SCHGDD of type $(3, 2^{w/2})$ on $B \times Z_w$ with group set $\{\{x\} \times Z_w : x \in B\}$ and hole set $\{B \times \{i, w/2 + i\} : 0 \leq i \leq w/2 - 1\}$. For the $B \in \mathcal{B}$ of size 5, construct a 3-SCHGDD of type $(5, 1^w)$ on $B \times Z_w$ with group set $\{\{x\} \times Z_w : x \in B\}$ and hole set $\{B \times \{i\} : 0 \leq i \leq w - 1\}$. Note that the required 3-SCHGDDs here can be found in Theorem 3.5. Let \mathcal{A}_B be a set of base blocks for each $B \in \mathcal{B}$. Let $\mathcal{A}_1 = \cup_{B \in \mathcal{B}} \mathcal{A}_B$. Clearly, $|\mathcal{A}_1| = (55w - 100)/3$.

By Example 5.1, there exists a 3-SCGDP* of type $2^{(11,5)}$ on $I_{11} \times Z_2$ with group set $\{\{i\} \times Z_2 : i \in I_{11}\}$, which has 32 base blocks. Suppose \mathcal{F} be a family of base blocks of this design. For each $B \in \mathcal{F}$, let $B' = \{(a, wx/2) : (a, x) \in B\}$. Let $\mathcal{A}_2 = \cup_{B \in \mathcal{F}} B'$. It is readily checked that $\mathcal{A}_1 \cup \mathcal{A}_2$ forms a family of base blocks of the required 3-SCGDP of type w^{11} with $J^*(11 \times 1 \times w, 3, 1) = (55w - 4)/3$ base blocks. \square

Lemma 5.3 *Let $u \equiv 11 \pmod{12}$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1, 5 \pmod{6}$ and $w \equiv 10 \pmod{12}$.*

Proof When $u = 11$, by Lemma 4.7, there exists a w -cyclic 3-IGDD of type $(vw, w)^{11}$. Filling in the hole with a 3-SCGDP of type w^{11} with $J^*(11 \times 1 \times w, 3, 1)$ base blocks from Lemma 5.2, by Construction 4.1, we obtain a w -cyclic 3-GDP of type $(vw)^{11}$ with $J^*(11 \times v \times w, 3, 1)$ base blocks.

When $u \geq 23$, by Lemma 4.26, there exists a w -cyclic 3-IGDD of type $(vw)^{(u,11)}$. Filling in the hole with a w -cyclic 3-GDP of type $(vw)^{11}$ with $J^*(11 \times v \times w, 3, 1)$ base blocks, by Construction 4.17, we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

5.2 The case of $u \equiv 2 \pmod{6}$

We first deal with the cases of $w \equiv 0, 1 \pmod{6}$.

Lemma 5.4 *Suppose $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1 \pmod{2}$ and $w \equiv 6 \pmod{12}$.*

Proof When $w = 6$, start with a 6-cyclic 3-IGDD of type $6^{(u,6)}$ from Lemma 4.28. Fill in the hole with a 3-SCGDP of type 6^6 with $J^*(6 \times 1 \times 6, 3, 1)$ base blocks, which exists by Corollary 2.6. Then we obtain a 3-SCGDP of type 6^u with $J^*(u \times 1 \times 6, 3, 1)$ base blocks. Further apply Construction 4.1 with a 6-cyclic 3-IGDD of type $(6v, 6)^u$ for any $v \equiv 1 \pmod{2}$ and $v \geq 3$, which can be found in Lemma 4.13, then we obtain a 6-cyclic 3-GDP of type $(6v)^u$ with $J^*(u \times v \times 6, 3, 1)$ base blocks.

When $w \geq 18$, by Lemma 3.10, there is a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (6v)^{w/6})$. Fill in the holes with above 6-cyclic 3-GDP of type $(6v)^u$ with $J^*(u \times v \times 6, 3, 1)$. We then obtain the required design by Construction 3.3. \square

Lemma 5.5 *Suppose $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \not\equiv 0 \pmod{6}$ and $w \equiv 1 \pmod{6}$.*

Proof When $w = 1$, an optimal 1-cyclic 3-GDP of type v^u with $J^*(u \times v \times 1, 3, 1)$ base blocks can be found in Corollary 2.6.

When $w \geq 7$, by Lemma 3.10, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type (u, v^w) . Then apply Construction 3.3 with above 1-cyclic 3-GDP of type v^u to obtain the required design. \square

We next deal with the case of $w \equiv 2 \pmod{6}$.

Lemma 5.6 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 3 \pmod{6}$ and $w \equiv 2 \pmod{12}$.*

Proof (1) $w = 2$. When $v = 3$, by Corollary 4.29, there exists a 2-cyclic 3-IGDD of type $6^{(u,6)}$. Fill in the hole with a 2-cyclic 3-GDP of type 6^6 from Corollary 2.6. By Construction 4.17, we obtain a 2-cyclic 3-GDP of type 6^u with $J^*(u \times 3 \times 2, 3, 1)$ base blocks.

When $v \equiv 3 \pmod{6}$ and $v \geq 9$, by Corollary 4.14, there exists a 2-cyclic 3-IGDD of type $(2v, 6)^u$. Fill in the hole with above 2-cyclic 3-GDP of type 6^u to obtain a 2-cyclic 3-GDP of type $(2v)^u$ with $J^*(u \times v \times 2, 3, 1)$ base blocks.

(2) $w \geq 14$. By Lemma 3.11, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (2v)^{w/2})$. Apply Construction 3.3 with a 2-cyclic 3-GDP of type $(2v)^u$ with $J^*(u \times v \times 2, 3, 1)$ base blocks to obtain the required designs. \square

Lemma 5.7 *Suppose $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1, 2 \pmod{3}$ and $w \equiv 2 \pmod{6}$.*

Proof (1) $u = 8$. When $v = 1$, the required 3-SCGDPs of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks are constructed directly in Lemmas C.1 and C.5 in Supporting Information. When $v \geq 2$, there exists a w -cyclic 3-IGDD of type $(vw, w)^8$ by Lemma 4.9. Fill in the hole with above 3-SCGDP of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks to obtain the required design.

(2) $u \equiv 2 \pmod{12}$ and $v \equiv 1, 5 \pmod{6}$. We first deal with the case of $(v, w) = (1, 2)$. When $u = 14$, by Lemma B.2 in Supporting Information, there exists a 2-cyclic 3-IGDD of type $2^{(14,2)}$, which is also a 3-SCGDP of type 2^{14} with $J^*(14 \times 1 \times 2, 3, 1)$ base blocks. When $u = 26$, by Lemma B.5 in Supporting Information, there exists a 2-cyclic 3-IGDD of type $2^{(26,11)}$. Fill in the hole with a 3-SCGDP of type 2^{11} with $J^*(11 \times 1 \times 2, 3, 1)$ base blocks from Corollary 2.6 to obtain a 3-SCGDP of type 2^{26} with $J^*(26 \times 1 \times 2, 3, 1)$ base blocks. When $u \geq 38$, by Lemma 4.25, there exists a 2-cyclic 3-IGDD of type $2^{(u,14)}$. Fill in the hole with above 3-SCGDP of type 2^{14} to obtain a 3-SCGDP of type 2^u with $J^*(u \times 1 \times 2, 3, 1)$ base blocks.

When $v \geq 5$ and $w = 2$, by Lemma 4.11, there exists a 2-cyclic 3-IGDD of type $(2v, 2)^u$. Filling in the hole with a 3-SCGDP of type 2^u with $J^*(u \times 1 \times 2, 3, 1)$ base blocks, by Construction 4.1, we obtain a 2-cyclic 3-GDP of type $(2v)^u$ with $J^*(u \times v \times 2, 3, 1)$ base blocks.

When $w \geq 8$, a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (2v)^{w/2})$ exists by Lemma 3.10. Fill in the holes with above 2-cyclic 3-GDP of type $(2v)^u$ with $J^*(u \times v \times 2, 3, 1)$ base blocks. Then we obtain the required design by Construction 3.3.

(3) Other values. By Lemma 4.30, there exists a w -cyclic 3-IGDD of type $(vw)^{(u,5)}$. Take a w -cyclic 3-GDP of type $(vw)^5$ with $J^*(5 \times v \times w, 3, 1)$ base blocks from Corollary 2.6 to fill in the hole. We then obtain the required w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Below we deal with the case of $w \equiv 3 \pmod{6}$.

Lemma 5.8 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv w \equiv 3 \pmod{6}$.*

Proof By Lemma 3.13, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type (u, v^w) . Filling in the hole with a 1-cyclic 3-GDP of type v^u with $J^*(u \times v \times 1, 3, 1)$ base blocks from Corollary 2.6, we then obtain the required design by Construction 3.3. \square

Lemma 5.9 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1, 5 \pmod{6}$ and $w \equiv 3 \pmod{6}$.*

Proof (1) $w = 3$ and $u \in \{8, 14\}$. For $v \in \{1, 5\}$, Lemmas C.3 and C.4 in Supporting Information provide the required designs. For $v \geq 7$, let $v = 6t + s$, where $s \in \{1, 5\}$. By Lemma 4.15, there exists a 3-cyclic 3-IGDP of type $(3v, 3s)^u$, which has $u(v - s)(3(u - 1)(v + s) - 1)/6$ base blocks. Filling in the hole with above 3-cyclic 3-GDP of type $(3s)^u$ to obtain the required designs.

(2) $w = 3$ and $u \geq 20$. By Lemma 4.31, there exists a 3-cyclic 3-IGDP of type $(3v)^{(u,8)}$ with $v(u - 8)(3(u + 7)v - 1)/6$ base blocks. Fill in the hole with a 3-cyclic 3-GDP of type $(3v)^8$ with $J^*(8 \times v \times 3, 3, 1)$ base blocks. By Construction 4.17, we obtain a 3-cyclic 3-GDP of type $(3v)^u$ with $J^*(u \times v \times 3, 3, 1)$ base blocks.

(3) $w \geq 9$. By Lemma 3.10, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (3v)^{w/3})$. Filling in the hole with above 3-cyclic 3-GDP of type $(3v)^u$ with $J^*(u \times v \times 3, 3, 1)$ base blocks, by Construction 3.3, we obtain the required design. \square

For dealing with the case of $w \equiv 4 \pmod{6}$, we need to construct more 3-SCGDP*s.

Lemma 5.10 *There exists a 3-SCGDP* of type $2^{(u,5)}$ with $(u(u-1)-14)/3$ base blocks for any $u \equiv 2 \pmod{12}$ and $u \geq 14$.*

Proof When $u = 14$, we construct the required design in Lemma C.2 in Supporting Information. When $u = 26$, by Lemma B.5 in Supporting Information, there exists a 2-cyclic 3-IGDD of type $2^{(26,11)}$. Take a 3-SCGDP* of type $2^{(11,5)}$ with 32 base blocks from Example 5.1. Then fill in the hole to obtain a 3-SCGDP* of type $2^{(26,5)}$ with 212 base blocks. Note that the 2-cyclic 3-IGDD of type $2^{(26,11)}$ has 180 base blocks.

When $u \geq 38$, by Lemma 4.25, there exists a 2-cyclic 3-IGDD of type $2^{(u,14)}$. Filling in the hole with above 3-SCGDP* of type $2^{(14,5)}$ with 56 base blocks, then we obtain a 3-SCGDP* of type $2^{(u,5)}$ with $(u(u-1)-14)/3$ base blocks. Note that the 2-cyclic 3-IGDD of type $2^{(u,14)}$ has $(u(u-1)-182)/3$ base blocks. \square

Lemma 5.11 *Suppose $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1, 2 \pmod{3}$ and $w \equiv 4 \pmod{12}$.*

Proof When $w = 4$, by Lemma 4.27, there exists a 4-cyclic 3-IGDD of type $4^{(u,2)}$, which is also a 3-SCGDP of type 4^u with $J^*(u \times 1 \times 4, 3, 1)$ base blocks. For $v \geq 2$, by Lemma 4.10, there exists a 4-cyclic 3-IGDD of type $(4v, 4)^u$. Filling in the hole with above 3-SCGDP of type 4^u , by Construction 4.1, we obtain a 4-cyclic 3-GDP of type $(4v)^u$ with $J^*(u \times v \times 4, 3, 1)$ base blocks.

For $w \geq 16$, by Lemma 3.10, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (4v)^{w/4})$. Filling in the holes with a 4-cyclic 3-GDP of type $(4v)^u$ with $J^*(u \times v \times 4, 3, 1)$ base blocks, by Construction 3.3, we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Lemma 5.12 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1, 2 \pmod{3}$ and $w \equiv 10 \pmod{12}$.*

Proof (1) $u = 8$. When $v = 1$, there exists a 3-SCGDP of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks by Lemma C.6 in Supporting Information. When $v \geq 2$, by Lemma 4.9, there exists a w -cyclic 3-IGDD of type $(vw, w)^8$. Filling in the hole with above 3-SCGDP of type w^8 , we obtain a w -cyclic 3-GDP of type $(vw)^8$ with $J^*(8 \times v \times w, 3, 1)$ base blocks.

(2) $u \equiv 2 \pmod{12}$ and $v \equiv 1, 5 \pmod{6}$. For $v = 1$, the proof is similar to that of Lemma 5.2. Here we start from a PBD($u, \{3, 4, 5^*\}$). Give weight w to each point of this PBD. For each block of size k , $k \in \{3, 4\}$, construct a 3-SCHGDD of type $(k, 2^{w/2})$. For the unique block of size 5, construct a 3-SCHGDD of type $(5, 1^w)$. Note that the required 3-SCHGDDs of type $(k, 2^{w/2})$ for $k \in \{3, 4\}$ and 3-SCGDP*s of type $2^{(u,5)}$ with $(u(u-1)-14)/3$ base blocks exist by Theorem 3.5 and Lemma 5.10, respectively.

When $v \geq 5$, start with a w -cyclic 3-IGDD of type $(vw, w)^u$ from Lemma 4.12, and fill in the hole with above 3-SCGDP of type w^u with $J^*(u \times 1 \times w, 3, 1)$ base blocks. Then we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks.

(3) Other values. By Lemma 4.30, there exists a w -cyclic 3-IGDD of type $(vw)^{(u,5)}$. Filling in the hole with a w -cyclic 3-GDP of type $(vw)^5$ with $J^*(5 \times v \times w, 3, 1)$ base blocks from Corollary 2.6, we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Lemma 5.13 *Let $u \equiv 2 \pmod{12}$ and $u \geq 14$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 3 \pmod{6}$ and $w \equiv 10 \pmod{12}$.*

Proof By Lemma 3.11, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type $(u, (2v)^{w/2})$. Apply Construction 3.3 with a 2-cyclic 3-GDP of type $(2v)^u$ with $J^*(u \times v \times 2, 3, 1)$ base blocks, which exists by Lemma 5.6. Then we obtain the required w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Finally, we deal with the case of $w \equiv 5 \pmod{6}$.

Lemma 5.14 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 2, 4 \pmod{6}$ and $w \equiv 5 \pmod{6}$.*

Proof By Lemma 2.1, there exists a 3-GDD of type $3^{2(u-2)/3}5^1$, which is also a PBD($2u+1, \{3, 5^*\}$). Delete one point which not belonging to the block of size 5, we obtain a $\{3, 5^*\}$ -GDD of type 2^u . Give weight $vw/2$ and input w -cyclic 3-GDDs of type $(vw/2)^3$ and a w -cyclic 3-GDP of type $(vw/2)^5$ with $J^*(5 \times v/2 \times w, 3, 1)$ base blocks from Corollary 2.6. Then we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Lemma 5.15 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 5 \pmod{6}$ and $w \equiv 5 \pmod{6}$.*

Proof By Theorem 3.5, there exists a 3-SCHGDD of type $(u, 1^{vw})$. Fill in the holes with a 1-cyclic 3-GDP of type 1^u with $u(u-2)/6$ blocks, which is also a 3-GDD of type $2^{u/2}$ and exists by Lemma 2.1. We then obtain a 3-SCGDP of type $(vw)^u$ with $J^*(u \times 1 \times vw, 3, 1)$ base blocks. Then, by Lemma 2.4, we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $vJ^*(u \times 1 \times vw, 3, 1) = J^*(u \times v \times w, 3, 1)$ base blocks. \square

Lemma 5.16 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 3 \pmod{6}$ and $w \equiv 5 \pmod{6}$.*

Proof By Lemma 3.11, there exists a $(v, w, 1)$ -cyclic 3-HGDD of type (u, v^w) . Apply Construction 3.3 with a 1-cyclic 3-GDP of type v^u with $J^*(u \times v \times 1, 3, 1)$ base blocks, which exists by Corollary 2.6. Then we obtain a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks. \square

Lemma 5.17 *Let $u \equiv 2 \pmod{6}$ and $u \geq 8$. Then there exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any $v \equiv 1 \pmod{6}$ and $w \equiv 5 \pmod{6}$.*

Proof (1) $u \in \{8, 14\}$. When $v = 1$, Lemmas C.7 and C.8 in Supporting Information provide the required designs. When $v \geq 7$, by Lemma 4.16, there exists a w -cyclic 3-IGDP of type $(vw, w)^u$, which has $u(v-1)((u-1)(v+1)w-1)/6$ base blocks. Filling in the hole using a 3-SCGDP of type w^u with $J^*(u \times 1 \times w, 3, 1)$ base blocks, we then obtain the required designs by Construction 4.1.

(2) $u \geq 20$. By Lemma 4.31, there exists a w -cyclic 3-IGDP of type $(vw)^{(u,8)}$ which has $v(u-8)((u+7)vw-1)/6$ base blocks. Filling in the hole with a w -cyclic 3-GDP of type $(vw)^8$ with $J^*(8 \times v \times w, 3, 1)$ base blocks, we obtain the required design by Construction 4.17. \square

Now, we are in the position to prove our main result of this section.

Theorem 5.18 *There exists a w -cyclic 3-GDP of type $(vw)^u$ with $J^*(u \times v \times w, 3, 1)$ base blocks for any positive integers v, w and $u \geq 3$.*

Proof Combining Lemmas 5.3-5.9 and 5.11-5.17, and Corollary 2.6, the conclusion then follows. \square

6 Concluding remarks

By Theorems 2.5 and 5.18, the size of an optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOC is finally determined.

Theorem 6.1 *There exists an optimal AM-OPP 3-D $(u \times v \times w, 3, 1)$ -OOC with $J^*(u \times v \times w, 3, 1)$ codewords for any positive integers v, w and $u \geq 3$.*

In [22], Wang and Yin proved that an SCHP $(2, k, uw)$ of type w^u , which is in fact a k -SCGDP of type w^u , is equivalent to an AM-OPP 2-D $(u \times w, k, 1)$ -OOC. Therefore the size of an optimal AM-OPP 2-D $(u \times w, 3, 1)$ -OOCs is also determined by Theorem 5.18.

Theorem 6.2 *There exists an optimal AM-OPP 2-D $(u \times w, 3, 1)$ -OOC with $J^*(u \times 1 \times w, 3, 1)$ codewords for any positive integers w and $u \geq 3$.*

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Supporting Information

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A h -cyclic 3-IGDDs of type $(gh, th)^u$

Lemma A.1 *There exists an h -perfect $2h$ -cyclic 3-IGDD of type $(4h, 2h)^8$ for $h \in \{1, 2, 4\}$.*

Proof Let $X = I \times I_2 \times Z_{2h}$, $Y = I \times \{0\} \times Z_{2h}$, and $\mathcal{G} = \{\{i\} \times I_2 \times Z_{2h} : i \in I\}$, where $I = Z_7 \cup \{\infty\}$. All base blocks can be obtained by developing following initial base blocks by $(+1, -, -) \bmod (7, -, -)$, where $\infty + 1 = \infty$.

$h = 1 :$

$$\begin{array}{lll} \{(0, 0, 0), (1, 1, 0), (3, 1, 1)\}, & \{(0, 0, 0), (5, 1, 0), (2, 1, 0)\}, & \{(0, 0, 0), (6, 1, 1), (\infty, 1, 1)\}, \\ \{(0, 0, 0), (6, 1, 0), (4, 1, 0)\}, & \{(0, 0, 0), (1, 1, 1), (2, 1, 1)\}, & \{(0, 0, 0), (5, 1, 1), (\infty, 1, 0)\}, \\ \{(0, 0, 0), (4, 1, 1), (3, 1, 0)\}, & \{(1, 1, 0), (4, 1, 1), (\infty, 0, 0)\}. \end{array}$$

$h = 2 :$

$$\begin{array}{lll} \{(0, 0, 0), (1, 1, 0), (3, 1, 1)\}, & \{(0, 0, 0), (5, 1, 2), (2, 1, 0)\}, & \{(0, 0, 0), (4, 1, 1), (3, 1, 0)\}, \\ \{(0, 0, 0), (6, 1, 2), (2, 1, 3)\}, & \{(0, 0, 0), (1, 1, 3), (6, 1, 1)\}, & \{(0, 0, 0), (5, 1, 1), (2, 1, 2)\}, \\ \{(0, 0, 0), (4, 1, 0), (3, 1, 2)\}, & \{(0, 0, 0), (1, 1, 2), (4, 1, 2)\}, & \{(0, 0, 0), (3, 1, 3), (5, 1, 3)\}, \\ \{(0, 0, 0), (5, 1, 0), (6, 1, 0)\}, & \{(0, 0, 0), (4, 1, 3), (\infty, 1, 0)\}, & \{(0, 0, 0), (1, 1, 1), (\infty, 1, 1)\}, \\ \{(0, 0, 0), (2, 1, 1), (\infty, 1, 3)\}, & \{(0, 0, 0), (6, 1, 3), (\infty, 1, 2)\}, & \{(1, 1, 0), (2, 1, 3), (\infty, 0, 0)\}, \\ \{(1, 1, 0), (6, 1, 1), (\infty, 0, 3)\}. \end{array}$$

$h = 4 :$

$$\begin{array}{lll} \{(0, 0, 0), (1, 1, 0), (3, 1, 1)\}, & \{(0, 0, 0), (5, 1, 2), (2, 1, 4)\}, & \{(0, 0, 0), (4, 1, 5), (3, 1, 0)\}, \\ \{(0, 0, 0), (6, 1, 6), (2, 1, 3)\}, & \{(0, 0, 0), (1, 1, 7), (6, 1, 5)\}, & \{(0, 0, 0), (5, 1, 1), (2, 1, 2)\}, \\ \{(0, 0, 0), (4, 1, 4), (5, 1, 0)\}, & \{(0, 0, 0), (1, 1, 6), (3, 1, 6)\}, & \{(0, 0, 0), (3, 1, 7), (2, 1, 1)\}, \\ \{(0, 0, 0), (4, 1, 3), (5, 1, 6)\}, & \{(0, 0, 0), (6, 1, 4), (2, 1, 0)\}, & \{(0, 0, 0), (1, 1, 5), (6, 1, 2)\}, \\ \{(0, 0, 0), (5, 1, 7), (2, 1, 7)\}, & \{(0, 0, 0), (4, 1, 2), (5, 1, 4)\}, & \{(0, 0, 0), (6, 1, 3), (1, 1, 1)\}, \\ \{(0, 0, 0), (1, 1, 4), (5, 1, 3)\}, & \{(0, 0, 0), (3, 1, 5), (2, 1, 5)\}, & \{(0, 0, 0), (4, 1, 1), (6, 1, 0)\}, \\ \{(0, 0, 0), (1, 1, 3), (\infty, 1, 1)\}, & \{(0, 0, 0), (3, 1, 4), (6, 1, 7)\}, & \{(0, 0, 0), (5, 1, 5), (4, 1, 6)\}, \\ \{(0, 0, 0), (4, 1, 0), (\infty, 1, 3)\}, & \{(0, 0, 0), (6, 1, 1), (\infty, 1, 5)\}, & \{(0, 0, 0), (1, 1, 2), (\infty, 1, 2)\}, \\ \{(0, 0, 0), (3, 1, 3), (\infty, 1, 0)\}, & \{(0, 0, 0), (2, 1, 6), (\infty, 1, 7)\}, & \{(0, 0, 0), (4, 1, 7), (\infty, 1, 6)\}, \\ \{(0, 0, 0), (3, 1, 2), (\infty, 1, 4)\}, & \{(1, 1, 0), (6, 1, 4), (\infty, 0, 6)\}, & \{(1, 1, 0), (2, 1, 1), (\infty, 0, 4)\}, \\ \{(1, 1, 0), (4, 1, 2), (\infty, 0, 1)\}, & \{(1, 1, 0), (6, 1, 3), (\infty, 0, 0)\}. \end{array}$$

Lemma A.2 *There exists a 4-cyclic 3-IGDD of type $(8, 4)^u$ for any $u \in \{4, 6, 14\}$.*

Proof Let $X = I \times I_2 \times Z_4$, $Y = I \times \{0\} \times Z_4$, and $\mathcal{G} = \{\{i\} \times I_2 \times Z_4 : i \in I\}$, where $I = Z_{u-1} \cup \{\infty\}$. All base blocks can be obtained by developing following initial base blocks by $(+1, -, -) \bmod (u-1, -, -)$, where $\infty + 1 = \infty$.

$u = 4 :$

$$\begin{array}{lll} \{(0,0,0), (1,1,0), (2,1,2)\}, & \{(0,0,0), (1,1,3), (2,1,3)\}, & \{(0,0,0), (2,1,1), (\infty,1,0)\}, \\ \{(0,0,0), (1,1,2), (\infty,1,2)\}, & \{(0,0,0), (2,1,0), (\infty,1,1)\}, & \{(0,0,0), (1,1,1), (\infty,1,3)\}, \\ \{(1,1,0), (0,1,1), (\infty,0,0)\}, & \{(1,1,0), (2,1,1), (\infty,0,2)\}. \end{array}$$

$u = 6 :$

$$\begin{array}{lll} \{(0,0,0), (1,1,0), (3,1,1)\}, & \{(0,0,0), (2,1,3), (1,1,1)\}, & \{(0,0,0), (4,1,0), (2,1,0)\}, \\ \{(0,0,0), (3,1,2), (4,1,1)\}, & \{(0,0,0), (1,1,2), (3,1,0)\}, & \{(0,0,0), (3,1,3), (4,1,3)\}, \\ \{(0,0,0), (2,1,1), (\infty,1,0)\}, & \{(0,0,0), (4,1,2), (\infty,1,2)\}, & \{(0,0,0), (1,1,3), (\infty,1,1)\}, \\ \{(0,0,0), (2,1,2), (\infty,1,3)\}, & \{(1,1,0), (0,1,3), (\infty,0,0)\}, & \{(1,1,0), (4,1,1), (\infty,0,3)\}. \end{array}$$

$u = 14 :$

$$\begin{array}{lll} \{(0,0,0), (1,1,0), (3,1,1)\}, & \{(0,0,0), (5,1,2), (8,1,3)\}, & \{(0,0,0), (7,1,3), (\infty,1,1)\}, \\ \{(0,0,0), (9,1,0), (8,1,0)\}, & \{(0,0,0), (11,1,1), (1,1,3)\}, & \{(0,0,0), (2,1,3), (7,1,0)\}, \\ \{(0,0,0), (4,1,0), (9,1,2)\}, & \{(0,0,0), (6,1,1), (2,1,2)\}, & \{(0,0,0), (8,1,2), (12,1,3)\}, \\ \{(0,0,0), (10,1,3), (11,1,0)\}, & \{(0,0,0), (12,1,0), (4,1,3)\}, & \{(0,0,0), (1,1,1), (12,1,1)\}, \\ \{(0,0,0), (3,1,2), (\infty,1,2)\}, & \{(0,0,0), (5,1,3), (4,1,1)\}, & \{(0,0,0), (9,1,1), (\infty,1,0)\}, \\ \{(0,0,0), (11,1,2), (4,1,2)\}, & \{(0,0,0), (2,1,0), (6,1,0)\}, & \{(0,0,0), (6,1,2), (8,1,1)\}, \\ \{(0,0,0), (10,1,0), (7,1,1)\}, & \{(0,0,0), (1,1,2), (3,1,0)\}, & \{(0,0,0), (3,1,3), (11,1,3)\}, \\ \{(0,0,0), (5,1,0), (12,1,2)\}, & \{(0,0,0), (2,1,1), (5,1,1)\}, & \{(0,0,0), (6,1,3), (10,1,1)\}, \\ \{(0,0,0), (7,1,2), (\infty,1,3)\}, & \{(0,0,0), (9,1,3), (10,1,2)\}, & \{(1,1,0), (7,1,3), (\infty,0,0)\}, \\ \{(1,1,0), (8,1,3), (\infty,0,2)\}. \end{array}$$

Lemma A.3 *Let $s \in \{1, 5\}$ and $u \in \{8, 14\}$. Then there exists a $\{3, u-2\}$ -IGDD of type $(6+s, s)^u$, where the blocks of size $u-2$ form a partition of the points outside the hole.*

Proof When $s = 1$, let $X = Z_u \times I$, $Y = Z_u \times \{\infty\}$ and $\mathcal{G} = \{\{i\} \times I : i \in Z_u\}$, where $I = I_6 \cup \{\infty\}$. When $s = 5$, let $X = Z_u \times I$, $Y = Z_u \times \{a, b, c, d, e\}$ and $\mathcal{G} = \{\{i\} \times I : i \in Z_u\}$, where $I = I_6 \cup \{a, b, c, d, e\}$. All blocks can be obtained by developing base blocks by $(+1, -) \bmod (u, -)$. Note that the lengths of orbits generated by the blocks of size $u-2$ are $u/2$.

(1) Base blocks of size $u-2$.

$u = 8 :$

$$\{(0,0), (1,2), (2,4), (4,0), (5,2), (6,4)\}, \{(0,1), (1,3), (2,5), (4,1), (5,3), (6,5)\}.$$

$u = 14 :$

$$\{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (7,0), (8,1), (9,2), (10,3), (11,4), (12,5)\}.$$

(2) Base blocks of size 3.

$(u, s) = (8, 1) :$

$$\begin{array}{llll}
\{(0,0), (1,0), (2,1)\}, & \{(0,0), (1,3), (2,0)\}, & \{(0,0), (1,4), (2,2)\}, & \{(0,0), (1,5), (2,3)\}, \\
\{(0,0), (2,5), (3,0)\}, & \{(0,0), (3,1), (4,1)\}, & \{(0,0), (3,2), (4,2)\}, & \{(0,0), (3,3), (4,3)\}, \\
\{(0,0), (3,4), (4,4)\}, & \{(0,0), (3,5), (4,5)\}, & \{(0,0), (5,1), (6,2)\}, & \{(0,0), (5,3), (6,1)\}, \\
\{(0,0), (5,4), (1,\infty)\}, & \{(0,0), (5,5), (2,\infty)\}, & \{(0,0), (6,3), (3,\infty)\}, & \{(0,0), (6,5), (7,\infty)\}, \\
\{(0,0), (7,1), (4,\infty)\}, & \{(0,0), (7,2), (5,\infty)\}, & \{(0,0), (7,4), (6,\infty)\}, & \{(0,1), (1,4), (2,1)\}, \\
\{(0,1), (1,5), (2,2)\}, & \{(0,1), (2,3), (3,2)\}, & \{(0,1), (2,4), (3,3)\}, & \{(0,1), (3,1), (6,4)\}, \\
\{(0,1), (3,5), (4,4)\}, & \{(0,1), (4,2), (5,5)\}, & \{(0,1), (4,3), (5,4)\}, & \{(0,1), (4,5), (2,\infty)\}, \\
\{(0,1), (5,2), (1,\infty)\}, & \{(0,1), (6,2), (7,\infty)\}, & \{(0,1), (6,3), (4,\infty)\}, & \{(0,1), (7,2), (6,\infty)\}, \\
\{(0,1), (7,5), (3,\infty)\}, & \{(0,2), (1,3), (3,2)\}, & \{(0,2), (2,2), (4,3)\}, & \{(0,2), (2,4), (4,4)\}, \\
\{(0,2), (2,5), (5,5)\}, & \{(0,2), (3,3), (5,3)\}, & \{(0,2), (3,4), (5,\infty)\}, & \{(0,2), (3,5), (2,\infty)\}, \\
\{(0,2), (4,5), (6,5)\}, & \{(0,2), (6,4), (3,\infty)\}, & \{(0,3), (2,4), (3,\infty)\}, & \{(0,3), (2,5), (6,4)\}, \\
\{(0,3), (3,3), (4,\infty)\}, & \{(0,3), (3,4), (6,5)\}, & \{(0,3), (3,5), (5,4)\}, & \{(0,3), (4,4), (2,\infty)\}, \\
\{(0,3), (4,5), (7,\infty)\}, & \{(0,4), (1,5), (3,\infty)\}, & \{(0,4), (2,5), (5,4)\}.
\end{array}$$

$$(u, s) = (8, 5) :$$

$$\begin{array}{llll}
\{(1,0), (2,0), (0,a)\}, & \{(1,1), (2,1), (0,a)\}, & \{(1,2), (2,2), (0,a)\}, & \{(1,3), (2,3), (0,a)\}, \\
\{(1,4), (2,4), (0,a)\}, & \{(1,5), (2,5), (0,a)\}, & \{(3,0), (4,3), (0,a)\}, & \{(3,1), (4,0), (0,a)\}, \\
\{(3,2), (4,1), (0,a)\}, & \{(3,3), (4,2), (0,a)\}, & \{(3,4), (5,1), (0,a)\}, & \{(3,5), (4,4), (0,a)\}, \\
\{(4,5), (6,0), (0,a)\}, & \{(5,0), (7,5), (0,a)\}, & \{(5,2), (7,3), (0,a)\}, & \{(5,3), (6,1), (0,a)\}, \\
\{(5,4), (6,2), (0,a)\}, & \{(5,5), (7,4), (0,a)\}, & \{(6,3), (7,0), (0,a)\}, & \{(6,4), (7,1), (0,a)\}, \\
\{(6,5), (7,2), (0,a)\}, & \{(1,0), (2,4), (0,b)\}, & \{(1,1), (2,5), (0,b)\}, & \{(1,2), (2,0), (0,b)\}, \\
\{(1,3), (3,0), (0,b)\}, & \{(1,4), (2,3), (0,b)\}, & \{(1,5), (2,1), (0,b)\}, & \{(2,2), (3,3), (0,b)\}, \\
\{(3,1), (4,4), (0,b)\}, & \{(3,2), (4,5), (0,b)\}, & \{(3,4), (6,3), (0,b)\}, & \{(3,5), (6,2), (0,b)\}, \\
\{(4,0), (7,0), (0,b)\}, & \{(4,1), (6,0), (0,b)\}, & \{(4,2), (6,1), (0,b)\}, & \{(4,3), (6,4), (0,b)\}, \\
\{(5,0), (6,5), (0,b)\}, & \{(5,1), (7,1), (0,b)\}, & \{(5,2), (7,2), (0,b)\}, & \{(5,3), (7,3), (0,b)\}, \\
\{(5,4), (7,4), (0,b)\}, & \{(5,5), (7,5), (0,b)\}, & \{(1,0), (3,0), (0,c)\}, & \{(1,1), (3,2), (0,c)\}, \\
\{(1,2), (3,4), (0,c)\}, & \{(1,3), (3,1), (0,c)\}, & \{(1,4), (2,0), (0,c)\}, & \{(1,5), (2,3), (0,c)\}, \\
\{(2,1), (4,3), (0,c)\}, & \{(2,2), (6,1), (0,c)\}, & \{(2,4), (4,2), (0,c)\}, & \{(2,5), (5,0), (0,c)\}, \\
\{(3,3), (6,2), (0,c)\}, & \{(3,5), (6,4), (0,c)\}, & \{(4,0), (6,3), (0,c)\}, & \{(4,1), (7,0), (0,c)\}, \\
\{(4,4), (6,5), (0,c)\}, & \{(4,5), (7,1), (0,c)\}, & \{(5,1), (7,4), (0,c)\}, & \{(5,2), (7,5), (0,c)\}, \\
\{(5,3), (7,2), (0,c)\}, & \{(5,4), (7,3), (0,c)\}, & \{(5,5), (6,0), (0,c)\}, & \{(1,0), (3,1), (0,d)\}, \\
\{(1,1), (4,1), (0,d)\}, & \{(1,2), (3,0), (0,d)\}, & \{(1,3), (5,0), (0,d)\}, & \{(1,4), (5,1), (0,d)\}, \\
\{(1,5), (6,0), (0,d)\}, & \{(2,0), (5,2), (0,d)\}, & \{(2,1), (5,3), (0,d)\}, & \{(2,2), (5,4), (0,d)\}, \\
\{(2,3), (5,5), (0,d)\}, & \{(2,4), (6,2), (0,d)\}, & \{(2,5), (6,1), (0,d)\}, & \{(3,2), (6,3), (0,d)\}, \\
\{(3,3), (6,4), (0,d)\}, & \{(3,4), (6,5), (0,d)\}, & \{(3,5), (7,0), (0,d)\}, & \{(4,0), (7,1), (0,d)\}, \\
\{(4,2), (7,2), (0,d)\}, & \{(4,3), (7,3), (0,d)\}, & \{(4,4), (7,4), (0,d)\}, & \{(4,5), (7,5), (0,d)\}, \\
\{(1,0), (4,3), (0,e)\}, & \{(1,1), (5,3), (0,e)\}, & \{(1,2), (5,5), (0,e)\}, & \{(1,3), (4,0), (0,e)\}, \\
\{(1,4), (6,0), (0,e)\}, & \{(1,5), (5,4), (0,e)\}, & \{(2,0), (6,1), (0,e)\}, & \{(2,1), (5,2), (0,e)\}, \\
\{(2,2), (5,1), (0,e)\}, & \{(2,3), (6,2), (0,e)\}, & \{(2,4), (5,0), (0,e)\}, & \{(2,5), (4,2), (0,e)\}, \\
\{(3,0), (7,2), (0,e)\}, & \{(3,1), (6,4), (0,e)\}, & \{(3,2), (6,5), (0,e)\}, & \{(3,3), (7,4), (0,e)\}, \\
\{(3,4), (7,0), (0,e)\}, & \{(3,5), (7,3), (0,e)\}, & \{(4,1), (7,5), (0,e)\}, & \{(4,4), (7,1), (0,e)\}, \\
\{(4,5), (6,3), (0,e)\}, & \{(0,0), (1,1), (2,2)\}, & \{(1,3), (2,4), (3,5)\}.
\end{array}$$

$$(u, s) = (14, 1) :$$

$\{(0, 0), (1, 0), (2, 3)\},$	$\{(0, 0), (1, 2), (8, 3)\},$	$\{(0, 0), (1, 4), (8, 5)\},$	$\{(0, 0), (1, 5), (12, 1)\},$
$\{(0, 0), (2, 0), (8, 4)\},$	$\{(0, 0), (2, 1), (12, 2)\},$	$\{(0, 0), (2, 4), (10, 1)\},$	$\{(0, 0), (2, 5), (4, 5)\},$
$\{(0, 0), (3, 0), (12, 3)\},$	$\{(0, 0), (3, 1), (5, 1)\},$	$\{(0, 0), (3, 2), (13, 4)\},$	$\{(0, 0), (3, 4), (7, 3)\},$
$\{(0, 0), (3, 5), (7, 1)\},$	$\{(0, 0), (4, 0), (13, 1)\},$	$\{(0, 0), (4, 1), (10, \infty)\},$	$\{(0, 0), (4, 2), (5, \infty)\},$
$\{(0, 0), (4, 3), (5, 0)\},$	$\{(0, 0), (5, 2), (4, \infty)\},$	$\{(0, 0), (5, 3), (9, 4)\},$	$\{(0, 0), (5, 4), (13, 2)\},$
$\{(0, 0), (6, 0), (3, \infty)\},$	$\{(0, 0), (6, 1), (7, 5)\},$	$\{(0, 0), (6, 2), (12, \infty)\},$	$\{(0, 0), (6, 3), (1, \infty)\},$
$\{(0, 0), (6, 5), (10, 4)\},$	$\{(0, 0), (7, 2), (13, 5)\},$	$\{(0, 0), (7, 4), (2, \infty)\},$	$\{(0, 0), (8, 2), (10, 5)\},$
$\{(0, 0), (9, 5), (6, \infty)\},$	$\{(0, 0), (10, 2), (11, 1)\},$	$\{(0, 0), (11, 2), (9, \infty)\},$	$\{(0, 0), (11, 3), (8, \infty)\},$
$\{(0, 0), (11, 5), (7, \infty)\},$	$\{(0, 0), (12, 4), (13, \infty)\},$	$\{(0, 1), (1, 1), (11, 3)\},$	$\{(0, 1), (1, 3), (7, 3)\},$
$\{(0, 1), (1, 4), (7, 4)\},$	$\{(0, 1), (2, 2), (9, 4)\},$	$\{(0, 1), (2, 4), (5, 2)\},$	$\{(0, 1), (2, 5), (7, 2)\},$
$\{(0, 1), (3, 1), (8, 1)\},$	$\{(0, 1), (3, 2), (5, \infty)\},$	$\{(0, 1), (3, 3), (4, 2)\},$	$\{(0, 1), (4, 1), (12, 3)\},$
$\{(0, 1), (4, 3), (13, 3)\},$	$\{(0, 1), (4, 4), (5, 3)\},$	$\{(0, 1), (5, 4), (7, 5)\},$	$\{(0, 1), (5, 5), (11, 2)\},$
$\{(0, 1), (6, 2), (13, 5)\},$	$\{(0, 1), (6, 3), (4, \infty)\},$	$\{(0, 1), (6, 5), (7, \infty)\},$	$\{(0, 1), (8, 4), (10, \infty)\},$
$\{(0, 1), (8, 5), (13, \infty)\},$	$\{(0, 1), (9, 2), (3, \infty)\},$	$\{(0, 1), (9, 5), (11, \infty)\},$	$\{(0, 1), (11, 4), (9, \infty)\},$
$\{(0, 1), (12, 2), (2, \infty)\},$	$\{(0, 1), (12, 4), (8, \infty)\},$	$\{(0, 1), (12, 5), (1, \infty)\},$	$\{(0, 1), (13, 4), (12, \infty)\},$
$\{(0, 2), (1, 2), (10, 2)\},$	$\{(0, 2), (1, 4), (4, 3)\},$	$\{(0, 2), (1, 5), (11, 2)\},$	$\{(0, 2), (2, 2), (13, 5)\},$
$\{(0, 2), (2, 3), (12, 3)\},$	$\{(0, 2), (3, 3), (10, \infty)\},$	$\{(0, 2), (3, 4), (12, 5)\},$	$\{(0, 2), (4, 4), (13, 4)\},$
$\{(0, 2), (5, 3), (12, 4)\},$	$\{(0, 2), (5, 4), (9, \infty)\},$	$\{(0, 2), (5, 5), (9, 3)\},$	$\{(0, 2), (6, 2), (3, \infty)\},$
$\{(0, 2), (6, 3), (7, \infty)\},$	$\{(0, 2), (8, 4), (5, \infty)\},$	$\{(0, 2), (10, 3), (11, 3)\},$	$\{(0, 3), (1, 5), (11, 5)\},$
$\{(0, 3), (2, 3), (7, 5)\},$	$\{(0, 3), (2, 4), (5, \infty)\},$	$\{(0, 3), (3, 3), (6, \infty)\},$	$\{(0, 3), (3, 4), (8, 5)\},$
$\{(0, 3), (3, 5), (6, 5)\},$	$\{(0, 3), (4, 5), (2, \infty)\},$	$\{(0, 3), (5, 4), (10, \infty)\},$	$\{(0, 3), (6, 4), (13, \infty)\},$
$\{(0, 3), (9, 4), (12, 5)\},$	$\{(0, 3), (12, 4), (4, \infty)\},$	$\{(0, 3), (13, 5), (8, \infty)\},$	$\{(0, 4), (1, 4), (11, 4)\},$
$\{(0, 4), (2, 4), (13, 5)\},$	$\{(0, 4), (4, 5), (8, \infty)\},$	$\{(0, 4), (6, 5), (12, 5)\},$	$\{(0, 5), (1, 5), (7, \infty)\},$
$\{(0, 5), (5, 5), (13, \infty)\}.$			

$(u, s) = (14, 5) :$

$\{(1, 0), (2, 0), (0, a)\},$	$\{(1, 1), (8, 0), (0, a)\},$	$\{(1, 2), (13, 1), (0, a)\},$	$\{(1, 3), (7, 1), (0, a)\},$
$\{(1, 4), (8, 2), (0, a)\},$	$\{(1, 5), (4, 4), (0, a)\},$	$\{(2, 1), (5, 0), (0, a)\},$	$\{(2, 2), (11, 2), (0, a)\},$
$\{(2, 3), (9, 5), (0, a)\},$	$\{(2, 4), (6, 2), (0, a)\},$	$\{(2, 5), (3, 4), (0, a)\},$	$\{(3, 0), (5, 1), (0, a)\},$
$\{(3, 1), (7, 4), (0, a)\},$	$\{(3, 2), (11, 5), (0, a)\},$	$\{(3, 3), (12, 4), (0, a)\},$	$\{(3, 5), (6, 3), (0, a)\},$
$\{(4, 0), (10, 4), (0, a)\},$	$\{(4, 1), (13, 0), (0, a)\},$	$\{(4, 2), (11, 0), (0, a)\},$	$\{(4, 3), (11, 1), (0, a)\},$
$\{(4, 5), (10, 3), (0, a)\},$	$\{(5, 2), (13, 2), (0, a)\},$	$\{(5, 3), (10, 5), (0, a)\},$	$\{(5, 4), (9, 3), (0, a)\},$
$\{(5, 5), (6, 5), (0, a)\},$	$\{(6, 0), (13, 5), (0, a)\},$	$\{(6, 1), (9, 1), (0, a)\},$	$\{(6, 4), (8, 3), (0, a)\},$
$\{(7, 0), (11, 3), (0, a)\},$	$\{(7, 2), (13, 3), (0, a)\},$	$\{(7, 3), (10, 2), (0, a)\},$	$\{(7, 5), (12, 0), (0, a)\},$
$\{(8, 1), (12, 1), (0, a)\},$	$\{(8, 4), (10, 1), (0, a)\},$	$\{(8, 5), (12, 3), (0, a)\},$	$\{(9, 0), (12, 2), (0, a)\},$
$\{(9, 2), (13, 4), (0, a)\},$	$\{(9, 4), (12, 5), (0, a)\},$	$\{(10, 0), (11, 4), (0, a)\},$	$\{(1, 0), (4, 4), (0, b)\},$
$\{(1, 1), (5, 0), (0, b)\},$	$\{(1, 2), (8, 5), (0, b)\},$	$\{(1, 3), (11, 2), (0, b)\},$	$\{(1, 4), (11, 3), (0, b)\},$
$\{(1, 5), (11, 4), (0, b)\},$	$\{(2, 0), (13, 5), (0, b)\},$	$\{(2, 1), (5, 5), (0, b)\},$	$\{(2, 2), (5, 2), (0, b)\},$
$\{(2, 3), (3, 0), (0, b)\},$	$\{(2, 4), (4, 2), (0, b)\},$	$\{(2, 5), (12, 0), (0, b)\},$	$\{(3, 1), (4, 1), (0, b)\},$
$\{(3, 2), (5, 1), (0, b)\},$	$\{(3, 3), (9, 0), (0, b)\},$	$\{(3, 4), (6, 4), (0, b)\},$	$\{(3, 5), (5, 4), (0, b)\},$
$\{(4, 0), (9, 2), (0, b)\},$	$\{(4, 3), (12, 1), (0, b)\},$	$\{(4, 5), (6, 1), (0, b)\},$	$\{(5, 3), (6, 2), (0, b)\},$
$\{(6, 0), (7, 3), (0, b)\},$	$\{(6, 3), (9, 1), (0, b)\},$	$\{(6, 5), (12, 5), (0, b)\},$	$\{(7, 0), (12, 4), (0, b)\},$
$\{(7, 1), (9, 4), (0, b)\},$	$\{(7, 2), (12, 3), (0, b)\},$	$\{(7, 4), (13, 4), (0, b)\},$	$\{(7, 5), (10, 5), (0, b)\},$
$\{(8, 0), (12, 2), (0, b)\},$	$\{(8, 1), (9, 3), (0, b)\},$	$\{(8, 2), (11, 1), (0, b)\},$	$\{(8, 3), (11, 0), (0, b)\},$
$\{(8, 4), (13, 1), (0, b)\},$	$\{(9, 5), (10, 3), (0, b)\},$	$\{(10, 0), (11, 5), (0, b)\},$	$\{(10, 1), (13, 2), (0, b)\},$
$\{(10, 2), (13, 0), (0, b)\},$	$\{(10, 4), (13, 3), (0, b)\},$	$\{(1, 0), (13, 0), (0, c)\},$	$\{(1, 1), (3, 1), (0, c)\},$
$\{(1, 2), (5, 1), (0, c)\},$	$\{(1, 3), (10, 1), (0, c)\},$	$\{(1, 4), (12, 3), (0, c)\},$	$\{(1, 5), (13, 2), (0, c)\},$
$\{(2, 0), (8, 1), (0, c)\},$	$\{(2, 1), (6, 2), (0, c)\},$	$\{(2, 2), (6, 5), (0, c)\},$	$\{(2, 3), (13, 1), (0, c)\},$

$\{(2, 4), (12, 5), (0, c)\},$	$\{(2, 5), (13, 3), (0, c)\},$	$\{(3, 0), (11, 5), (0, c)\},$	$\{(3, 2), (7, 0), (0, c)\},$
$\{(3, 3), (7, 1), (0, c)\},$	$\{(3, 4), (8, 4), (0, c)\},$	$\{(3, 5), (13, 5), (0, c)\},$	$\{(4, 0), (10, 5), (0, c)\},$
$\{(4, 1), (8, 3), (0, c)\},$	$\{(4, 2), (11, 3), (0, c)\},$	$\{(4, 3), (9, 4), (0, c)\},$	$\{(4, 4), (10, 0), (0, c)\},$
$\{(4, 5), (12, 1), (0, c)\},$	$\{(5, 0), (11, 2), (0, c)\},$	$\{(5, 2), (9, 2), (0, c)\},$	$\{(5, 3), (9, 3), (0, c)\},$
$\{(5, 4), (12, 0), (0, c)\},$	$\{(5, 5), (10, 4), (0, c)\},$	$\{(6, 0), (9, 5), (0, c)\},$	$\{(6, 1), (13, 4), (0, c)\},$
$\{(6, 3), (12, 4), (0, c)\},$	$\{(6, 4), (9, 1), (0, c)\},$	$\{(7, 2), (8, 5), (0, c)\},$	$\{(7, 3), (9, 0), (0, c)\},$
$\{(7, 4), (11, 4), (0, c)\},$	$\{(7, 5), (11, 1), (0, c)\},$	$\{(8, 0), (11, 0), (0, c)\},$	$\{(8, 2), (10, 3), (0, c)\},$
$\{(10, 2), (12, 2), (0, c)\},$	$\{(1, 0), (7, 0), (0, d)\},$	$\{(1, 1), (2, 5), (0, d)\},$	$\{(1, 2), (12, 4), (0, d)\},$
$\{(1, 3), (4, 3), (0, d)\},$	$\{(1, 4), (3, 0), (0, d)\},$	$\{(1, 5), (13, 4), (0, d)\},$	$\{(2, 0), (7, 3), (0, d)\},$
$\{(2, 1), (7, 2), (0, d)\},$	$\{(2, 2), (10, 1), (0, d)\},$	$\{(2, 3), (9, 4), (0, d)\},$	$\{(2, 4), (8, 5), (0, d)\},$
$\{(3, 1), (8, 0), (0, d)\},$	$\{(3, 2), (12, 3), (0, d)\},$	$\{(3, 3), (5, 4), (0, d)\},$	$\{(3, 4), (12, 2), (0, d)\},$
$\{(3, 5), (11, 2), (0, d)\},$	$\{(4, 0), (13, 0), (0, d)\},$	$\{(4, 1), (13, 5), (0, d)\},$	$\{(4, 2), (5, 1), (0, d)\},$
$\{(4, 4), (9, 0), (0, d)\},$	$\{(4, 5), (11, 4), (0, d)\},$	$\{(5, 0), (8, 1), (0, d)\},$	$\{(5, 2), (8, 4), (0, d)\},$
$\{(5, 3), (10, 3), (0, d)\},$	$\{(5, 5), (11, 1), (0, d)\},$	$\{(6, 0), (8, 3), (0, d)\},$	$\{(6, 1), (13, 2), (0, d)\},$
$\{(6, 2), (12, 0), (0, d)\},$	$\{(6, 3), (12, 5), (0, d)\},$	$\{(6, 4), (12, 1), (0, d)\},$	$\{(6, 5), (11, 5), (0, d)\},$
$\{(7, 1), (13, 1), (0, d)\},$	$\{(7, 4), (11, 0), (0, d)\},$	$\{(7, 5), (9, 5), (0, d)\},$	$\{(8, 2), (9, 2), (0, d)\},$
$\{(9, 1), (10, 0), (0, d)\},$	$\{(9, 3), (10, 5), (0, d)\},$	$\{(10, 2), (13, 3), (0, d)\},$	$\{(10, 4), (11, 3), (0, d)\},$
$\{(1, 0), (13, 1), (0, e)\},$	$\{(1, 1), (6, 1), (0, e)\},$	$\{(1, 2), (10, 5), (0, e)\},$	$\{(1, 3), (13, 3), (0, e)\},$
$\{(1, 4), (7, 2), (0, e)\},$	$\{(1, 5), (10, 1), (0, e)\},$	$\{(2, 0), (4, 5), (0, e)\},$	$\{(2, 1), (7, 4), (0, e)\},$
$\{(2, 2), (3, 0), (0, e)\},$	$\{(2, 3), (6, 2), (0, e)\},$	$\{(2, 4), (7, 5), (0, e)\},$	$\{(2, 5), (5, 2), (0, e)\},$
$\{(3, 1), (13, 0), (0, e)\},$	$\{(3, 2), (5, 0), (0, e)\},$	$\{(3, 3), (4, 1), (0, e)\},$	$\{(3, 4), (11, 1), (0, e)\},$
$\{(3, 5), (7, 0), (0, e)\},$	$\{(4, 0), (10, 3), (0, e)\},$	$\{(4, 2), (9, 1), (0, e)\},$	$\{(4, 3), (11, 0), (0, e)\},$
$\{(4, 4), (6, 4), (0, e)\},$	$\{(5, 1), (12, 5), (0, e)\},$	$\{(5, 3), (10, 0), (0, e)\},$	$\{(5, 4), (13, 2), (0, e)\},$
$\{(5, 5), (6, 0), (0, e)\},$	$\{(6, 3), (12, 3), (0, e)\},$	$\{(6, 5), (7, 1), (0, e)\},$	$\{(7, 3), (9, 2), (0, e)\},$
$\{(8, 0), (12, 0), (0, e)\},$	$\{(8, 1), (9, 4), (0, e)\},$	$\{(8, 2), (13, 5), (0, e)\},$	$\{(8, 3), (9, 3), (0, e)\},$
$\{(8, 4), (9, 0), (0, e)\},$	$\{(8, 5), (10, 2), (0, e)\},$	$\{(9, 5), (11, 3), (0, e)\},$	$\{(10, 4), (11, 2), (0, e)\},$
$\{(11, 4), (12, 1), (0, e)\},$	$\{(11, 5), (12, 2), (0, e)\},$	$\{(12, 4), (13, 4), (0, e)\},$	$\{(0, 0), (1, 2), (2, 4)\},$
$\{(1, 3), (3, 1), (5, 5)\}.$			

B h -cyclic 3-IGDDs of type $gh^{(u,t)}$

Lemma B.1 *Let $u \in \{20, 32, 44, 56\}$. Then there exists a 2-cyclic 3-IGDD of type $2^{(u,5)}$.*

Proof Suppose $I = Z_{u-5} \cup \{a, b, c, d, e\}$. Let $X = I \times Z_2$, $\mathcal{G} = \{\{i\} \times Z_2 : i \in I\}$, and $Y = \{a, b, c, d, e\} \times Z_2$. Base blocks are listed below. Developing these base blocks by $+(i, j) \bmod (u-5, 2)$ gives all blocks, where $(i, j) \in Z_{u-5} \times Z_2$ and a, b, c, d, e are fixed points. It is readily checked that each design is isomorphism to a 2-cyclic 3-IGDD of type $2^{(u,5)}$.

$u = 20 :$

$\{(0, 0), (1, 0), (3, 0)\},$	$\{(0, 0), (7, 0), (1, 1)\},$	$\{(0, 0), (3, 1), (b, 0)\},$	$\{(0, 0), (5, 1), (d, 0)\},$
$\{(0, 0), (4, 0), (9, 0)\},$	$\{(0, 0), (2, 1), (a, 0)\},$	$\{(0, 0), (4, 1), (c, 0)\},$	$\{(0, 0), (7, 1), (e, 0)\}.$

$u = 32 :$

$\{(0, 0), (1, 0), (3, 0)\},$	$\{(0, 0), (8, 0), (1, 1)\},$	$\{(0, 0), (12, 0), (3, 1)\},$	$\{(0, 0), (11, 1), (c, 0)\},$
$\{(0, 0), (4, 0), (9, 0)\},$	$\{(0, 0), (10, 0), (2, 1)\},$	$\{(0, 0), (4, 1), (a, 0)\},$	$\{(0, 0), (12, 1), (d, 0)\},$
$\{(0, 0), (6, 0), (13, 0)\},$	$\{(0, 0), (11, 0), (5, 1)\},$	$\{(0, 0), (10, 1), (b, 0)\},$	$\{(0, 0), (13, 1), (e, 0)\}.$

$u = 44 :$

$$\begin{array}{llll}
\{(0,0), (1,0), (3,0)\}, & \{(0,0), (11,0), (1,1)\}, & \{(0,0), (16,0), (21,1)\}, & \{(0,0), (14,1), (b,0)\}, \\
\{(0,0), (4,0), (9,0)\}, & \{(0,0), (12,0), (3,1)\}, & \{(0,0), (17,0), (23,1)\}, & \{(0,0), (15,1), (c,0)\}, \\
\{(0,0), (6,0), (13,0)\}, & \{(0,0), (14,0), (2,1)\}, & \{(0,0), (19,0), (26,1)\}, & \{(0,0), (17,1), (d,0)\}, \\
\{(0,0), (8,0), (18,0)\}, & \{(0,0), (15,0), (4,1)\}, & \{(0,0), (8,1), (a,0)\}, & \{(0,0), (19,1), (e,0)\}.
\end{array}$$

$u = 56 :$

$$\begin{array}{llll}
\{(0,0), (4,0), (46,0)\}, & \{(0,0), (10,0), (3,0)\}, & \{(0,0), (19,0), (b,0)\}, & \{(0,0), (5,0), (a,0)\}, \\
\{(0,0), (8,0), (25,1)\}, & \{(0,0), (11,1), (15,0)\}, & \{(0,0), (20,0), (22,1)\}, & \{(0,0), (16,0), (43,0)\}, \\
\{(0,0), (6,0), (17,0)\}, & \{(0,0), (12,0), (27,1)\}, & \{(0,0), (21,1), (9,0)\}, & \{(0,0), (20,1), (e,0)\}, \\
\{(0,0), (7,1), (25,0)\}, & \{(0,0), (13,1), (50,1)\}, & \{(0,0), (22,0), (16,1)\}, & \{(0,0), (21,0), (d,1)\}, \\
\{(0,0), (14,0), (32,0)\}, & \{(0,0), (1,1), (3,1)\}, & \{(0,0), (23,1), (10,1)\}, & \{(0,0), (23,0), (c,1)\}.
\end{array}$$

Lemma B.2 *There exists a 2-cyclic 3-IGDD of type $2^{(14,2)}$.*

Proof Let $X = (Z_6 \cup \{\infty\}) \times Z_4$, $\mathcal{G} = \{\{i\} \times \{j, 2+j\} : (i, j) \in Z_6 \times Z_2\} \cup \{\{\infty\} \times \{l, l+2\} : l \in Z_2\}$, and $Y = \{\infty\} \times Z_4$. We list 10 base blocks.

$$\begin{array}{llll}
\{(0,0), (1,0), (2,0)\}, & \{(0,0), (3,0), (0,1)\}, & \{(0,0), (1,1), (3,1)\}, & \{(0,0), (2,1), (1,2)\}, \\
\{(0,0), (4,1), (1,3)\}, & \{(0,0), (2,2), (\infty,0)\}, & \{(0,0), (5,2), (\infty,1)\}, & \{(0,0), (5,3), (\infty,3)\}, \\
\{(1,0), (1,1), (3,2)\}, & \{(1,0), (5,1), (\infty,2)\}.
\end{array}$$

Developing these base blocks by $+(2s, t) \bmod (6, 4)$ gives all blocks, where $(s, t) \in Z_3 \times Z_4$ and ∞ is a fixed point. It is readily checked that each design is isomorphism to a 2-cyclic 3-IGDD of type $2^{(u,2)}$. \square

Lemma B.3 *Let $u \in \{8, 14\}$. Then there exists a 4-cyclic 3-IGDD of type $4^{(u,2)}$.*

Proof Let $X = (Z_{(u-2)/2} \cup \{\infty\}) \times Z_8$, $\mathcal{G} = \{\{i\} \times \{j, 2+j, 4+j, 6+j\} : (i, j) \in Z_{(u-2)/2} \times Z_2\} \cup \{\{\infty\} \times \{l, 2+l, 4+l, 6+l\} : l \in Z_2\}$, and $Y = \{\infty\} \times Z_8$. Developing the following base blocks by $+(2s, t) \bmod ((u-2)/2, 8)$ gives all blocks, where $(s, t) \in Z_3 \times Z_8$ and ∞ is a fixed point. It is readily checked that each design is isomorphism to a 4-cyclic 3-IGDD of type $4^{(u,2)}$.

$u = 8 :$

$$\begin{array}{lll}
\{(0,0), (1,1), (0,3)\}, & \{(0,0), (1,4), (\infty,1)\}, & \{(0,0), (2,5), (1,5)\}, \\
\{(0,0), (1,7), (\infty,3)\}, & \{(0,0), (0,1), (\infty,7)\}, & \{(0,0), (1,2), (\infty,2)\}.
\end{array}$$

$u = 14 :$

$$\begin{array}{llll}
\{(0,0), (1,0), (2,0)\}, & \{(0,0), (3,0), (0,1)\}, & \{(0,0), (1,1), (3,1)\}, & \{(0,0), (2,1), (1,2)\}, \\
\{(0,0), (4,1), (0,3)\}, & \{(0,0), (3,2), (1,3)\}, & \{(0,0), (4,2), (3,4)\}, & \{(0,0), (2,3), (1,6)\}, \\
\{(0,0), (3,3), (2,4)\}, & \{(0,0), (4,3), (\infty,0)\}, & \{(0,0), (1,4), (1,5)\}, & \{(0,0), (5,4), (\infty,1)\}, \\
\{(0,0), (3,5), (\infty,3)\}, & \{(0,0), (5,5), (\infty,4)\}, & \{(0,0), (3,6), (\infty,6)\}, & \{(0,0), (5,6), (\infty,7)\}, \\
\{(0,0), (5,7), (\infty,2)\}, & \{(1,0), (3,1), (5,4)\}, & \{(1,0), (3,2), (1,5)\}, & \{(1,0), (5,2), (\infty,4)\}.
\end{array}$$

Lemma B.4 *There exists a 6-cyclic 3-IGDD of type $6^{(6,2)}$.*

Proof Let $X = Z_{36}$, $\mathcal{G} = \{i, 6+i, \dots, 30+i : 0 \leq i \leq 5\}$, $Y = \{x : x = 0, 1 \pmod{6}, x \in Z_{36}\}$. Developing the following base blocks by $+6 \bmod 36$ gives all blocks.

$$\begin{array}{ccccccc}
\{0, 2, 3\}, & \{0, 4, 5\}, & \{0, 8, 10\}, & \{0, 9, 11\}, & \{0, 14, 17\}, & \{0, 15, 16\}, & \{0, 20, 27\}, \\
\{0, 21, 26\}, & \{0, 22, 32\}, & \{0, 23, 33\}, & \{0, 28, 35\}, & \{0, 29, 34\}, & \{1, 2, 10\}, & \{1, 3, 11\}, \\
\{1, 4, 8\}, & \{1, 5, 26\}, & \{1, 9, 28\}, & \{1, 14, 33\}, & \{1, 15, 22\}, & \{1, 16, 35\}, & \{1, 17, 32\}, \\
\{1, 20, 29\}, & \{1, 21, 34\}, & \{1, 23, 27\}, & \{2, 15, 29\}, & \{2, 16, 27\}, & \{2, 22, 35\}, & \{3, 23, 34\}.
\end{array}$$

Lemma B.5 *There exists a 2-cyclic 3-IGDD of type $2^{(26,11)}$.*

Proof The required design is constructed on I_{52} with group set $\{\{i, 20 + i\} : 0 \leq i \leq 19\} \cup \{\{j, j + 6\} : 40 \leq j \leq 45\}$ and hole set $\{x : x \equiv 0 \pmod{4}, x < 40\} \cup \{y : y \geq 40\}$. The base blocks are listed as follows:

$$\begin{array}{ccccccc}
\{0, 1, 3\}, & \{0, 2, 10\}, & \{0, 5, 37\}, & \{0, 6, 39\}, & \{0, 7, 38\}, & \{0, 9, 19\}, \\
\{0, 11, 25\}, & \{0, 13, 22\}, & \{0, 14, 29\}, & \{0, 15, 31\}, & \{0, 17, 30\}, & \{0, 18, 34\}, \\
\{0, 21, 26\}, & \{0, 23, 35\}, & \{0, 27, 33\}, & \{1, 2, 38\}, & \{1, 5, 40\}, & \{1, 7, 47\}, \\
\{1, 13, 42\}, & \{1, 15, 17\}, & \{1, 18, 51\}, & \{1, 19, 45\}, & \{1, 22, 49\}, & \{1, 23, 50\}, \\
\{1, 30, 43\}, & \{1, 31, 44\}, & \{1, 34, 41\}, & \{2, 3, 40\}, & \{2, 7, 15\}, & \{2, 14, 44\}, \\
\{2, 19, 46\}, & \{2, 23, 42\}, & \{2, 27, 47\}, & \{2, 31, 45\}, & \{2, 39, 48\}, & \{3, 7, 43\}.
\end{array}$$

Let $\alpha = \prod_{i=0}^3(i \ 4 + i \ \cdots \ 36 + i) \prod_{i=0}^6(40 + i \ 46 + i)$ be a permutation on I_{52} . Let G be the group generated by α . All blocks are obtained by developing the base blocks under the action of G . \square

C w -cyclic 3-GDPs of type $(vw)^u$

Lemma C.1 *There exists a 3-SCGDP of type 2^8 with $J^*(8 \times 1 \times 2, 3, 1)$ base blocks.*

Proof Let $X = I_8 \times Z_2$, $\mathcal{G} = \{\{i\} \times Z_2 : i \in I_8\}$. Only the $J^*(8 \times 1 \times 2, 3, 1)$ base blocks are listed below.

$$\begin{array}{cccc}
\{(0, 0), (1, 0), (2, 0)\}, & \{(0, 0), (3, 0), (4, 0)\}, & \{(0, 0), (5, 0), (6, 0)\}, & \{(0, 0), (7, 0), (1, 1)\}, \\
\{(0, 0), (2, 1), (3, 1)\}, & \{(0, 0), (4, 1), (5, 1)\}, & \{(0, 0), (6, 1), (7, 1)\}, & \{(1, 0), (3, 0), (5, 0)\}, \\
\{(1, 0), (4, 0), (6, 0)\}, & \{(1, 0), (7, 0), (4, 1)\}, & \{(1, 0), (2, 1), (5, 1)\}, & \{(1, 0), (3, 1), (6, 1)\}, \\
\{(2, 0), (4, 0), (3, 1)\}, & \{(2, 0), (6, 0), (4, 1)\}, & \{(2, 0), (7, 0), (6, 1)\}, & \{(2, 0), (5, 1), (7, 1)\}, \\
\{(3, 0), (7, 0), (5, 1)\}.
\end{array}$$

Lemma C.2 *There exists a 3-SCGDP* of type $2^{(u,5)}$ with $(u(u-1)-14)/3$ base blocks for any $u \in \{11, 14\}$.*

Proof Let $X = I_u \times Z_2$, $\mathcal{G} = \{\{i\} \times Z_2 : i \in I_u\}$. Only base blocks are listed below.

$u = 11$:

$$\begin{array}{cccc}
\{(0, 0), (1, 0), (2, 0)\}, & \{(0, 0), (3, 0), (4, 0)\}, & \{(0, 0), (5, 0), (6, 0)\}, & \{(0, 0), (7, 0), (8, 0)\}, \\
\{(0, 0), (9, 0), (10, 0)\}, & \{(0, 0), (5, 1), (7, 1)\}, & \{(0, 0), (6, 1), (9, 1)\}, & \{(0, 0), (8, 1), (10, 1)\}, \\
\{(1, 0), (3, 0), (5, 0)\}, & \{(1, 0), (4, 0), (6, 0)\}, & \{(1, 0), (7, 0), (9, 0)\}, & \{(1, 0), (8, 0), (5, 1)\}, \\
\{(1, 0), (10, 0), (6, 1)\}, & \{(1, 0), (7, 1), (10, 1)\}, & \{(1, 0), (8, 1), (9, 1)\}, & \{(2, 0), (3, 0), (7, 0)\}, \\
\{(2, 0), (4, 0), (8, 0)\}, & \{(2, 0), (5, 0), (9, 0)\}, & \{(2, 0), (6, 0), (7, 1)\}, & \{(2, 0), (10, 0), (9, 1)\}, \\
\{(2, 0), (5, 1), (8, 1)\}, & \{(2, 0), (6, 1), (10, 1)\}, & \{(3, 0), (6, 0), (8, 1)\}, & \{(3, 0), (8, 0), (9, 1)\}, \\
\{(3, 0), (9, 0), (6, 1)\}, & \{(3, 0), (10, 0), (7, 1)\}, & \{(3, 0), (5, 1), (10, 1)\}, & \{(4, 0), (5, 0), (10, 1)\}, \\
\{(4, 0), (7, 0), (9, 1)\}, & \{(4, 0), (9, 0), (5, 1)\}, & \{(4, 0), (10, 0), (8, 1)\}, & \{(4, 0), (6, 1), (7, 1)\}.
\end{array}$$

$u = 14$:

$$\begin{array}{llll}
\{(0,0), (1,0), (2,0)\}, & \{(0,0), (3,0), (5,0)\}, & \{(0,0), (4,0), (6,0)\}, & \{(0,0), (7,0), (9,0)\}, \\
\{(0,0), (8,0), (10,0)\}, & \{(0,0), (11,0), (13,0)\}, & \{(0,0), (12,0), (5,1)\}, & \{(0,0), (6,1), (8,1)\}, \\
\{(0,0), (7,1), (12,1)\}, & \{(0,0), (9,1), (11,1)\}, & \{(0,0), (10,1), (13,1)\}, & \{(1,0), (3,0), (6,0)\}, \\
\{(1,0), (4,0), (7,0)\}, & \{(1,0), (5,0), (8,0)\}, & \{(1,0), (9,0), (12,0)\}, & \{(1,0), (10,0), (5,1)\}, \\
\{(1,0), (11,0), (6,1)\}, & \{(1,0), (13,0), (7,1)\}, & \{(1,0), (8,1), (11,1)\}, & \{(1,0), (9,1), (13,1)\}, \\
\{(1,0), (10,1), (12,1)\}, & \{(2,0), (3,0), (7,0)\}, & \{(2,0), (4,0), (8,0)\}, & \{(2,0), (5,0), (9,0)\}, \\
\{(2,0), (6,0), (10,0)\}, & \{(2,0), (11,0), (5,1)\}, & \{(2,0), (12,0), (8,1)\}, & \{(2,0), (13,0), (9,1)\}, \\
\{(2,0), (6,1), (11,1)\}, & \{(2,0), (7,1), (10,1)\}, & \{(2,0), (12,1), (13,1)\}, & \{(3,0), (4,0), (9,0)\}, \\
\{(3,0), (8,0), (12,0)\}, & \{(3,0), (10,0), (11,0)\}, & \{(3,0), (13,0), (8,1)\}, & \{(3,0), (5,1), (6,1)\}, \\
\{(3,0), (7,1), (13,1)\}, & \{(3,0), (9,1), (10,1)\}, & \{(3,0), (11,1), (12,1)\}, & \{(4,0), (5,0), (8,1)\}, \\
\{(4,0), (10,0), (9,1)\}, & \{(4,0), (11,0), (10,1)\}, & \{(4,0), (12,0), (11,1)\}, & \{(4,0), (13,0), (12,1)\}, \\
\{(4,0), (5,1), (7,1)\}, & \{(4,0), (6,1), (13,1)\}, & \{(5,0), (10,0), (13,1)\}, & \{(5,0), (11,0), (7,1)\}, \\
\{(5,0), (12,0), (9,1)\}, & \{(5,0), (13,0), (6,1)\}, & \{(6,0), (7,0), (12,1)\}, & \{(6,0), (9,0), (7,1)\}, \\
\{(6,0), (12,0), (10,1)\}, & \{(6,0), (8,1), (9,1)\}, & \{(7,0), (8,0), (10,1)\}, & \{(7,0), (11,0), (8,1)\}.
\end{array}$$

Lemma C.3 *There exists a 3-SCGDP of type 3^u with $J^*(u \times 1 \times 3, 3, 1)$ base blocks for any $u \in \{8, 14\}$.*

Proof Let $X = I \times Z_3$ and $\mathcal{G} = \{\{i\} \times Z_3 : i \in I\}$, where $I = Z_{u-2} \cup \{a, b\}$. The total $J^*(u \times 1 \times 3, 3, 1) = (3u^2 - 4u - 4)/6$ base blocks can be obtained by two parts.

Part I: Developing $\{(0,0), ((u-2)/3,0), (2(u-2)/3,0)\}$ by $+(1, -) \bmod (u-2, -)$ can generate $(u-2)/3$ base blocks.

Part II: Developing the following u initial base blocks by $+(2, -) \bmod (u-2, -)$ can generate $u(u-2)/2$ base blocks. Here a, b are fixed points.

$u = 8 :$

$$\begin{array}{llll}
\{(0,0), (1,0), (2,1)\}, & \{(0,0), (5,0), (1,2)\}, & \{(0,0), (1,1), (a,0)\}, & \{(0,0), (3,1), (a,1)\}, \\
\{(0,0), (4,1), (b,1)\}, & \{(0,0), (5,1), (a,2)\}, & \{(0,0), (3,2), (b,2)\}, & \{(1,0), (3,1), (b,2)\}.
\end{array}$$

$u = 14 :$

$$\begin{array}{llll}
\{(0,0), (1,0), (2,0)\}, & \{(0,0), (3,0), (5,0)\}, & \{(0,0), (7,0), (1,1)\}, & \{(0,0), (9,0), (2,1)\}, \\
\{(0,0), (3,1), (1,2)\}, & \{(0,0), (4,1), (3,2)\}, & \{(0,0), (5,1), (6,2)\}, & \{(0,0), (7,1), (9,2)\}, \\
\{(0,0), (8,1), (a,0)\}, & \{(0,0), (9,1), (b,0)\}, & \{(0,0), (10,1), (b,2)\}, & \{(0,0), (5,2), (a,1)\}, \\
\{(1,0), (5,1), (a,1)\}, & \{(1,0), (9,1), (b,1)\}.
\end{array}$$

Lemma C.4 *There exists a 3-cyclic 3-GDP of type 15^u with $J^*(u \times 5 \times 3, 3, 1)$ base blocks for any $u \in \{8, 14\}$.*

Proof We construct the required designs directly. Let $X = I \times I_5 \times Z_3$ and $\mathcal{G} = \{\{i\} \times I_5 \times Z_3 : i \in I\}$, where $I = Z_{u-1} \cup \{\infty\}$. The $J^*(u \times 5 \times 3, 3, 1) = (75u^2 - 80u - 2)/6$ base blocks are divide into two parts. The first part consists of $(4u^2 - 5u - 6)/6$ base blocks:

$$\begin{array}{l}
\{(2i, 0, 0), (2i+1, 0, 0), (\infty, 0, 0)\}, \quad 1 \leq i \leq (u-2)/2, \\
\{(a_j, l, 0), (b_j, l, 0), (c_j, l, 0)\}, \quad 1 \leq l \leq 4, \quad 1 \leq j \leq u(u-2)/6.
\end{array}$$

Here, $\{\{a_j, b_j, c_j\} : 1 \leq j \leq u(u-2)/6\}$ are the blocks of a 3-GDD of type $2^{u/2}$ on I .

The second part consists of $(u-1)(71u-4)/6$ base blocks, which can be obtained by developing $(71u-4)/6$ initial base blocks by $+(1, -, -) \bmod (u-1, -, -)$, where $\infty + 1 = \infty$. We list these initial base blocks below.

$u = 8 :$

$\{(0, 0, 0), (2, 0, 0), (1, 1, 0)\},$	$\{(0, 0, 0), (3, 0, 0), (4, 0, 1)\},$	$\{(0, 0, 0), (2, 1, 0), (5, 4, 2)\},$
$\{(0, 0, 0), (3, 1, 0), (1, 3, 2)\},$	$\{(0, 0, 0), (4, 1, 0), (1, 4, 2)\},$	$\{(0, 0, 0), (5, 1, 0), (6, 2, 2)\},$
$\{(0, 0, 0), (\infty, 1, 0), (3, 2, 0)\},$	$\{(0, 0, 0), (1, 2, 0), (\infty, 1, 2)\},$	$\{(0, 0, 0), (2, 2, 0), (1, 2, 2)\},$
$\{(0, 0, 0), (4, 2, 0), (2, 3, 2)\},$	$\{(0, 0, 0), (5, 2, 0), (\infty, 0, 2)\},$	$\{(0, 0, 0), (6, 2, 0), (1, 1, 2)\},$
$\{(0, 0, 0), (\infty, 2, 0), (1, 3, 0)\},$	$\{(0, 0, 0), (2, 3, 0), (5, 1, 1)\},$	$\{(0, 0, 0), (3, 3, 0), (1, 4, 0)\},$
$\{(0, 0, 0), (4, 3, 0), (5, 0, 1)\},$	$\{(0, 0, 0), (5, 3, 0), (3, 1, 2)\},$	$\{(0, 0, 0), (6, 3, 0), (2, 2, 1)\},$
$\{(0, 0, 0), (\infty, 3, 0), (4, 3, 1)\},$	$\{(0, 0, 0), (2, 4, 0), (6, 4, 2)\},$	$\{(0, 0, 0), (3, 4, 0), (2, 2, 2)\},$
$\{(0, 0, 0), (4, 4, 0), (5, 3, 1)\},$	$\{(0, 0, 0), (5, 4, 0), (2, 1, 2)\},$	$\{(0, 0, 0), (6, 4, 0), (4, 2, 1)\},$
$\{(0, 0, 0), (\infty, 4, 0), (6, 2, 1)\},$	$\{(0, 0, 0), (2, 0, 1), (5, 1, 2)\},$	$\{(0, 0, 0), (3, 0, 1), (4, 1, 2)\},$
$\{(0, 0, 0), (6, 0, 1), (4, 2, 2)\},$	$\{(0, 0, 0), (\infty, 0, 1), (3, 2, 1)\},$	$\{(0, 0, 0), (2, 1, 1), (3, 4, 1)\},$
$\{(0, 0, 0), (4, 1, 1), (1, 4, 1)\},$	$\{(0, 0, 0), (6, 1, 1), (4, 3, 2)\},$	$\{(0, 0, 0), (\infty, 1, 1), (6, 4, 1)\},$
$\{(0, 0, 0), (1, 2, 1), (5, 3, 2)\},$	$\{(0, 0, 0), (\infty, 2, 1), (5, 2, 2)\},$	$\{(0, 0, 0), (1, 3, 1), (\infty, 3, 2)\},$
$\{(0, 0, 0), (2, 3, 1), (6, 1, 2)\},$	$\{(0, 0, 0), (3, 3, 1), (\infty, 2, 2)\},$	$\{(0, 0, 0), (6, 3, 1), (3, 3, 2)\},$
$\{(0, 0, 0), (\infty, 3, 1), (3, 2, 2)\},$	$\{(0, 0, 0), (2, 4, 1), (\infty, 4, 2)\},$	$\{(0, 0, 0), (4, 4, 1), (3, 4, 2)\},$
$\{(0, 0, 0), (5, 4, 1), (2, 4, 2)\},$	$\{(0, 0, 0), (\infty, 4, 1), (4, 4, 2)\},$	$\{(\infty, 0, 0), (0, 1, 0), (1, 3, 1)\},$
$\{(\infty, 0, 0), (0, 3, 0), (6, 3, 2)\},$	$\{(\infty, 0, 0), (0, 4, 0), (5, 4, 2)\},$	$\{(\infty, 0, 0), (0, 1, 1), (2, 2, 2)\},$
$\{(\infty, 0, 0), (0, 4, 1), (1, 1, 2)\},$	$\{(0, 1, 0), (1, 2, 0), (4, 3, 1)\},$	$\{(0, 1, 0), (2, 2, 0), (\infty, 4, 1)\},$
$\{(0, 1, 0), (3, 2, 0), (6, 4, 1)\},$	$\{(0, 1, 0), (4, 2, 0), (2, 2, 2)\},$	$\{(0, 1, 0), (5, 2, 0), (1, 2, 1)\},$
$\{(0, 1, 0), (6, 2, 0), (4, 4, 1)\},$	$\{(0, 1, 0), (\infty, 2, 0), (4, 1, 1)\},$	$\{(0, 1, 0), (1, 3, 0), (6, 2, 1)\},$
$\{(0, 1, 0), (2, 3, 0), (3, 2, 1)\},$	$\{(0, 1, 0), (3, 3, 0), (5, 4, 2)\},$	$\{(0, 1, 0), (4, 3, 0), (6, 4, 0)\},$
$\{(0, 1, 0), (5, 3, 0), (2, 1, 2)\},$	$\{(0, 1, 0), (6, 3, 0), (2, 4, 2)\},$	$\{(0, 1, 0), (\infty, 3, 0), (4, 2, 2)\},$
$\{(0, 1, 0), (2, 4, 0), (4, 1, 2)\},$	$\{(0, 1, 0), (3, 4, 0), (6, 3, 2)\},$	$\{(0, 1, 0), (5, 4, 0), (3, 2, 2)\},$
$\{(0, 1, 0), (\infty, 4, 0), (2, 3, 2)\},$	$\{(0, 1, 0), (1, 1, 1), (\infty, 3, 2)\},$	$\{(0, 1, 0), (2, 1, 1), (\infty, 1, 2)\},$
$\{(0, 1, 0), (6, 1, 1), (1, 4, 2)\},$	$\{(0, 1, 0), (4, 2, 1), (1, 4, 1)\},$	$\{(0, 1, 0), (\infty, 2, 1), (1, 3, 2)\},$
$\{(0, 1, 0), (6, 3, 1), (5, 2, 2)\},$	$\{(0, 1, 0), (6, 2, 2), (\infty, 4, 2)\},$	$\{(\infty, 1, 0), (0, 3, 0), (4, 3, 2)\},$
$\{(\infty, 1, 0), (0, 3, 1), (5, 4, 2)\},$	$\{(\infty, 1, 0), (0, 4, 1), (1, 2, 2)\},$	$\{(0, 2, 0), (1, 3, 0), (5, 4, 0)\},$
$\{(0, 2, 0), (2, 3, 0), (4, 4, 1)\},$	$\{(0, 2, 0), (3, 3, 0), (6, 4, 1)\},$	$\{(0, 2, 0), (4, 3, 0), (3, 2, 2)\},$
$\{(0, 2, 0), (5, 3, 0), (1, 4, 0)\},$	$\{(0, 2, 0), (6, 3, 0), (1, 2, 2)\},$	$\{(0, 2, 0), (\infty, 3, 0), (2, 4, 0)\},$
$\{(0, 2, 0), (3, 4, 0), (6, 3, 1)\},$	$\{(0, 2, 0), (6, 4, 0), (5, 2, 1)\},$	$\{(0, 2, 0), (\infty, 2, 1), (5, 4, 2)\},$
$\{(0, 2, 0), (2, 3, 1), (3, 4, 2)\},$	$\{(0, 2, 0), (3, 3, 2), (4, 4, 2)\},$	$\{(\infty, 2, 0), (0, 4, 0), (2, 4, 2)\},$
$\{(0, 3, 0), (6, 4, 0), (5, 3, 1)\},$	$\{(0, 3, 0), (\infty, 4, 0), (2, 3, 1)\},$	$\{(0, 3, 0), (6, 3, 1), (5, 4, 2)\},$
$\{(\infty, 3, 0), (0, 4, 1), (1, 4, 2)\}.$		

$u = 14 :$

$\{(0, 0, 0), (2, 0, 0), (5, 0, 0)\},$	$\{(0, 0, 0), (4, 0, 0), (3, 3, 0)\},$	$\{(0, 0, 0), (6, 0, 0), (7, 4, 2)\},$
$\{(0, 0, 0), (1, 1, 0), (7, 3, 0)\},$	$\{(0, 0, 0), (2, 1, 0), (1, 4, 0)\},$	$\{(0, 0, 0), (3, 1, 0), (6, 0, 1)\},$
$\{(0, 0, 0), (4, 1, 0), (6, 4, 2)\},$	$\{(0, 0, 0), (5, 1, 0), (3, 1, 2)\},$	$\{(0, 0, 0), (6, 1, 0), (7, 3, 2)\},$
$\{(0, 0, 0), (7, 1, 0), (5, 2, 1)\},$	$\{(0, 0, 0), (8, 1, 0), (1, 4, 1)\},$	$\{(0, 0, 0), (9, 1, 0), (2, 4, 0)\},$

$\{(0, 0, 0), (10, 1, 0), (1, 2, 0)\},$	$\{(0, 0, 0), (11, 1, 0), (10, 4, 2)\},$	$\{(0, 0, 0), (12, 1, 0), (11, 3, 0)\},$
$\{(0, 0, 0), (\infty, 1, 0), (6, 3, 1)\},$	$\{(0, 0, 0), (2, 2, 0), (3, 2, 1)\},$	$\{(0, 0, 0), (3, 2, 0), (6, 1, 1)\},$
$\{(0, 0, 0), (4, 2, 0), (\infty, 0, 1)\},$	$\{(0, 0, 0), (5, 2, 0), (8, 0, 2)\},$	$\{(0, 0, 0), (6, 2, 0), (2, 1, 2)\},$
$\{(0, 0, 0), (7, 2, 0), (6, 3, 2)\},$	$\{(0, 0, 0), (8, 2, 0), (10, 3, 1)\},$	$\{(0, 0, 0), (9, 2, 0), (8, 4, 0)\},$
$\{(0, 0, 0), (10, 2, 0), (12, 3, 2)\},$	$\{(0, 0, 0), (11, 2, 0), (9, 4, 1)\},$	$\{(0, 0, 0), (12, 2, 0), (4, 3, 1)\},$
$\{(0, 0, 0), (\infty, 2, 0), (4, 1, 1)\},$	$\{(0, 0, 0), (1, 3, 0), (8, 1, 1)\},$	$\{(0, 0, 0), (2, 3, 0), (\infty, 4, 0)\},$
$\{(0, 0, 0), (4, 3, 0), (1, 0, 1)\},$	$\{(0, 0, 0), (5, 3, 0), (8, 3, 2)\},$	$\{(0, 0, 0), (6, 3, 0), (3, 4, 1)\},$
$\{(0, 0, 0), (8, 3, 0), (12, 0, 1)\},$	$\{(0, 0, 0), (9, 3, 0), (4, 0, 2)\},$	$\{(0, 0, 0), (10, 3, 0), (1, 1, 1)\},$
$\{(0, 0, 0), (\infty, 3, 0), (1, 2, 1)\},$	$\{(0, 0, 0), (3, 4, 0), (2, 1, 1)\},$	$\{(0, 0, 0), (4, 4, 0), (5, 2, 2)\},$
$\{(0, 0, 0), (5, 4, 0), (\infty, 4, 2)\},$	$\{(0, 0, 0), (6, 4, 0), (3, 4, 2)\},$	$\{(0, 0, 0), (7, 4, 0), (2, 4, 2)\},$
$\{(0, 0, 0), (9, 4, 0), (\infty, 2, 2)\},$	$\{(0, 0, 0), (10, 4, 0), (\infty, 0, 2)\},$	$\{(0, 0, 0), (11, 4, 0), (7, 3, 1)\},$
$\{(0, 0, 0), (12, 4, 0), (1, 3, 2)\},$	$\{(0, 0, 0), (2, 0, 1), (1, 1, 2)\},$	$\{(0, 0, 0), (3, 0, 1), (12, 1, 2)\},$
$\{(0, 0, 0), (4, 0, 1), (6, 2, 2)\},$	$\{(0, 0, 0), (7, 0, 1), (4, 1, 2)\},$	$\{(0, 0, 0), (8, 0, 1), (6, 1, 2)\},$
$\{(0, 0, 0), (10, 0, 1), (3, 2, 2)\},$	$\{(0, 0, 0), (11, 0, 1), (\infty, 1, 2)\},$	$\{(0, 0, 0), (3, 1, 1), (4, 2, 1)\},$
$\{(0, 0, 0), (5, 1, 1), (7, 2, 1)\},$	$\{(0, 0, 0), (7, 1, 1), (\infty, 2, 1)\},$	$\{(0, 0, 0), (8, 2, 1), (9, 4, 2)\},$
$\{(0, 0, 0), (9, 2, 1), (2, 4, 1)\},$	$\{(0, 0, 0), (11, 2, 1), (8, 1, 2)\},$	$\{(0, 0, 0), (12, 2, 1), (8, 4, 1)\},$
$\{(0, 0, 0), (1, 3, 1), (4, 4, 1)\},$	$\{(0, 0, 0), (2, 3, 1), (12, 2, 2)\},$	$\{(0, 0, 0), (3, 3, 1), (4, 4, 2)\},$
$\{(0, 0, 0), (8, 3, 1), (5, 1, 2)\},$	$\{(0, 0, 0), (9, 3, 1), (10, 3, 2)\},$	$\{(0, 0, 0), (11, 3, 1), (5, 3, 2)\},$
$\{(0, 0, 0), (12, 3, 1), (11, 3, 2)\},$	$\{(0, 0, 0), (\infty, 3, 1), (11, 4, 2)\},$	$\{(0, 0, 0), (5, 4, 1), (10, 2, 2)\},$
$\{(0, 0, 0), (6, 4, 1), (9, 1, 2)\},$	$\{(0, 0, 0), (7, 4, 1), (11, 1, 2)\},$	$\{(0, 0, 0), (10, 4, 1), (4, 2, 2)\},$
$\{(0, 0, 0), (11, 4, 1), (\infty, 3, 2)\},$	$\{(0, 0, 0), (12, 4, 1), (11, 2, 2)\},$	$\{(0, 0, 0), (\infty, 4, 1), (7, 2, 2)\},$
$\{(0, 0, 0), (7, 1, 2), (8, 4, 2)\},$	$\{(0, 0, 0), (1, 2, 2), (5, 4, 2)\},$	$\{(0, 0, 0), (2, 2, 2), (12, 4, 2)\},$
$\{(0, 0, 0), (8, 2, 2), (4, 3, 2)\},$	$\{(0, 0, 0), (9, 2, 2), (2, 3, 2)\},$	$\{(\infty, 0, 0), (0, 1, 0), (8, 4, 2)\},$
$\{(\infty, 0, 0), (0, 2, 0), (1, 1, 1)\},$	$\{(\infty, 0, 0), (0, 3, 0), (5, 3, 2)\},$	$\{(\infty, 0, 0), (0, 4, 0), (3, 2, 1)\},$
$\{(\infty, 0, 0), (0, 3, 1), (1, 1, 2)\},$	$\{(0, 1, 0), (3, 2, 0), (1, 3, 0)\},$	$\{(0, 1, 0), (5, 2, 0), (\infty, 1, 2)\},$
$\{(0, 1, 0), (6, 2, 0), (4, 4, 0)\},$	$\{(0, 1, 0), (7, 2, 0), (3, 4, 2)\},$	$\{(0, 1, 0), (8, 2, 0), (12, 2, 1)\},$
$\{(0, 1, 0), (9, 2, 0), (10, 3, 2)\},$	$\{(0, 1, 0), (10, 2, 0), (\infty, 3, 0)\},$	$\{(0, 1, 0), (11, 2, 0), (4, 1, 1)\},$
$\{(0, 1, 0), (12, 2, 0), (4, 3, 2)\},$	$\{(0, 1, 0), (2, 3, 0), (8, 1, 2)\},$	$\{(0, 1, 0), (3, 3, 0), (2, 4, 0)\},$
$\{(0, 1, 0), (4, 3, 0), (\infty, 3, 2)\},$	$\{(0, 1, 0), (5, 3, 0), (1, 4, 1)\},$	$\{(0, 1, 0), (7, 3, 0), (9, 1, 1)\},$
$\{(0, 1, 0), (8, 3, 0), (10, 4, 1)\},$	$\{(0, 1, 0), (9, 3, 0), (6, 1, 2)\},$	$\{(0, 1, 0), (10, 3, 0), (12, 3, 1)\},$
$\{(0, 1, 0), (11, 3, 0), (\infty, 4, 1)\},$	$\{(0, 1, 0), (3, 4, 0), (1, 1, 2)\},$	$\{(0, 1, 0), (5, 4, 0), (2, 1, 2)\},$
$\{(0, 1, 0), (7, 4, 0), (11, 2, 2)\},$	$\{(0, 1, 0), (8, 4, 0), (2, 2, 2)\},$	$\{(0, 1, 0), (9, 4, 0), (4, 3, 1)\},$
$\{(0, 1, 0), (10, 4, 0), (3, 1, 2)\},$	$\{(0, 1, 0), (11, 4, 0), (8, 4, 1)\},$	$\{(0, 1, 0), (\infty, 4, 0), (8, 3, 1)\},$
$\{(0, 1, 0), (1, 1, 1), (4, 2, 2)\},$	$\{(0, 1, 0), (3, 1, 1), (5, 2, 2)\},$	$\{(0, 1, 0), (6, 1, 1), (8, 3, 2)\},$
$\{(0, 1, 0), (8, 1, 1), (5, 3, 2)\},$	$\{(0, 1, 0), (\infty, 1, 1), (1, 3, 1)\},$	$\{(0, 1, 0), (1, 2, 1), (8, 2, 2)\},$
$\{(0, 1, 0), (5, 2, 1), (7, 4, 2)\},$	$\{(0, 1, 0), (6, 2, 1), (11, 4, 2)\},$	$\{(0, 1, 0), (7, 2, 1), (9, 2, 2)\},$
$\{(0, 1, 0), (8, 2, 1), (1, 2, 2)\},$	$\{(0, 1, 0), (9, 2, 1), (6, 2, 2)\},$	$\{(0, 1, 0), (10, 2, 1), (4, 4, 2)\},$
$\{(0, 1, 0), (\infty, 2, 1), (11, 3, 1)\},$	$\{(0, 1, 0), (5, 3, 1), (9, 4, 1)\},$	$\{(0, 1, 0), (6, 3, 1), (2, 3, 2)\},$
$\{(0, 1, 0), (9, 3, 1), (7, 3, 2)\},$	$\{(0, 1, 0), (\infty, 3, 1), (5, 4, 1)\},$	$\{(0, 1, 0), (4, 4, 1), (5, 4, 2)\},$
$\{(0, 1, 0), (11, 4, 1), (\infty, 4, 2)\},$	$\{(0, 1, 0), (12, 4, 1), (6, 4, 2)\},$	$\{(\infty, 1, 0), (0, 2, 0), (5, 4, 0)\},$
$\{(\infty, 1, 0), (0, 4, 1), (11, 4, 2)\},$	$\{(\infty, 1, 0), (0, 2, 2), (5, 3, 2)\},$	$\{(0, 2, 0), (1, 3, 0), (3, 2, 1)\},$
$\{(0, 2, 0), (2, 3, 0), (12, 4, 2)\},$	$\{(0, 2, 0), (3, 3, 0), (6, 3, 1)\},$	$\{(0, 2, 0), (4, 3, 0), (11, 2, 1)\},$
$\{(0, 2, 0), (7, 3, 0), (4, 2, 2)\},$	$\{(0, 2, 0), (8, 3, 0), (2, 4, 2)\},$	$\{(0, 2, 0), (10, 3, 0), (3, 4, 1)\},$
$\{(0, 2, 0), (12, 3, 0), (\infty, 3, 1)\},$	$\{(0, 2, 0), (1, 4, 0), (12, 3, 1)\},$	$\{(0, 2, 0), (2, 4, 0), (4, 3, 1)\},$
$\{(0, 2, 0), (3, 4, 0), (5, 2, 1)\},$	$\{(0, 2, 0), (7, 4, 0), (3, 4, 2)\},$	$\{(0, 2, 0), (8, 4, 0), (7, 3, 1)\},$
$\{(0, 2, 0), (\infty, 4, 0), (5, 2, 2)\},$	$\{(0, 2, 0), (12, 2, 1), (10, 3, 2)\},$	$\{(0, 2, 0), (\infty, 2, 1), (9, 3, 2)\},$
$\{(0, 2, 0), (1, 3, 1), (5, 4, 2)\},$	$\{(0, 2, 0), (8, 3, 1), (7, 4, 2)\},$	$\{(0, 2, 0), (9, 3, 1), (\infty, 2, 2)\},$
$\{(0, 2, 0), (10, 3, 1), (8, 4, 1)\},$	$\{(0, 2, 0), (4, 4, 1), (8, 3, 2)\},$	$\{(0, 2, 0), (10, 4, 1), (4, 3, 2)\},$
$\{(0, 2, 0), (7, 3, 2), (4, 4, 2)\},$	$\{(\infty, 2, 0), (0, 4, 0), (1, 4, 2)\},$	$\{(0, 3, 0), (1, 4, 0), (8, 4, 2)\},$
$\{(0, 3, 0), (2, 4, 0), (8, 3, 2)\},$	$\{(0, 3, 0), (5, 4, 0), (6, 3, 1)\},$	$\{(0, 3, 0), (6, 4, 0), (8, 4, 1)\},$
$\{(0, 3, 0), (7, 4, 0), (4, 3, 1)\},$	$\{(0, 3, 0), (8, 4, 0), (3, 4, 1)\},$	$\{(0, 3, 0), (9, 4, 0), (5, 4, 1)\}.$

Lemma C.5 *There exists a 3-SCGDP of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks for any $w \equiv 2 \pmod{6}$ and $w \geq 8$.*

Proof Let $X = I \times Z_w$ and $\mathcal{G} = \{\{i\} \times Z_w : i \in I\}$, where $I = Z_7 \cup \{\infty\}$. We construct the required designs directly.

(1) $w = 12s + 2$ and $s \geq 1$. All $112s + 18$ are divided into two parts. The first part consists of following 46 base blocks:

$$\begin{array}{lll} \{(2, 0), (4, 1), (\infty, s)\}, & \{(1, 0), (2, 6s + 1), (5, 10s + 1)\}, & \{(2, 0), (3, 6s + 1), (6, 10s + 1)\}, \\ \{(0, 0), (5, s), (3, 4s)\}, & \{(0, 0), (5, 4s + 1), (6, 10s + 2)\}, & \{(0, 0), (6, 10s + 1), (1, 10s + 2)\}, \\ \{(0, 0), (6, 0), (\infty, s)\}, & \{(0, 0), (1, 4s + 1), (2, 9s + 2)\}, & \{(0, 0), (2, 4s + 1), (1, 6s + 1)\}, \\ \{(0, 0), (1, 0), (2, 0)\}, & \{(0, 0), (5, 6s + 1), (6, 8s + 1)\}, & \{(0, 0), (6, 6s + 1), (5, 8s + 1)\}, \\ \{(1, 0), (3, 0), (\infty, s)\}, & \{(1, 0), (6, 4s + 1), (4, 8s + 2)\}, & \{(0, 0), (4, 10s + 1), (5, 12s + 1)\}, \\ \{(1, 0), (6, 3s), (4, 4s)\}, & \{(3, 0), (4, 4s + 1), (6, 8s + 2)\}, & \{(3, 0), (4, 5s + 1), (5, 6s + 1)\}, \\ \{(4, 0), (5, 0), (\infty, s)\}, & \{(0, 0), (1, 2s + 1), (3, 8s + 2)\}, & \{(4, 0), (5, 5s + 1), (6, 6s + 1)\}, \\ \{(0, 0), (1, s), (6, 2s)\}, & \{(4, 0), (5, 4s + 1), (6, 9s + 2)\}, & \{(1, 0), (5, 8s + 2), (3, 9s + 2)\}, \\ \{(2, 0), (3, 0), (\infty, s - 1)\}, & \{(1, 0), (2, 4s + 1), (6, 8s + 1)\}, & \{(1, 0), (2, 2s + 1), (4, 2s + 1)\}, \\ \{(1, 0), (5, 4s), (3, 8s + 1)\}, & \{(2, 0), (3, 4s + 1), (4, 4s + 1)\}, & \{(2, 0), (3, s), (4, 11s + 2)\}, \\ \{(2, 0), (3, 2s), (5, 2s + 1)\}, & \{(2, 0), (5, 8s + 2), (6, 8s + 2)\}, & \{(3, 0), (4, 2s), (5, 4s + 1)\}, \\ \{(3, 0), (5, 0), (4, 6s + 1)\}, & \{(3, 0), (4, 2s + 1), (6, 2s + 1)\}, & \{(4, 0), (6, 1), (5, 10s + 2)\}, \\ \{(0, 0), (5, 0), (\infty, s - 1)\}, & \{(2, 0), (3, 5s + 1), (4, 6s + 1)\}, & \{(1, 0), (2, s), (3, 11s + 2)\}, \\ \{(0, 0), (4, 4s), (2, 8s + 1)\}, & \{(0, 0), (1, 5s + 1), (6, 11s + 2)\}, & \{(1, 0), (6, 0), (\infty, s - 1)\}, \\ \{(0, 0), (5, 3s), (6, 7s + 1)\}, & \{(0, 0), (4, 8s + 2), (2, 11s + 2)\}, & \{(0, 0), (1, 2s), (3, 2s + 1)\}, \\ \{(1, 0), (2, 2s), (3, 4s + 1)\}. \end{array}$$

The second part consists of $112s - 28$ base blocks. All base blocks can be obtained by developing following $16s - 4$ initial base blocks by $(+1, -) \bmod (7, -)$, where $\infty + 1 = \infty$.

$$\begin{array}{l} \{(0, 0), (3, 6s + 1), (\infty, 8s + 2)\}, \\ \{(0, 0), (3, 8s + 1 + 2i), (\infty, 4s + i)\}, i \in [0, s], \\ \{(0, 0), (3, 4s + 1 + 2i), (\infty, 6s + 1 + i)\}, i \in [0, s - 1], \\ \{(0, 0), (3, 6s + 2 + 2i), (\infty, 5s + 1 + i)\}, i \in [0, s - 1], \\ \{(0, 0), (3, 8s + 4 + 2i), (\infty, 8s + 3 + i)\}, i \in [0, s - 1], \\ \{(0, 0), (1, 10s + 1 - i), (3, 10s + 3 + i)\}, i \in [0, 4s - 1], \\ \{(0, 0), (2, 8s + 2 + i), (4, 6s + 2 + 2i)\}, i \in [0, 2s - 1] \setminus \{s\}, \\ \{(0, 0), (1, 2s + 1 + i), (2, 2i + 1)\}, i \in [1, 4s - 1] \setminus \{2s, 3s\}, \\ \{(0, 0), (3, 2s + 1 + 2i), (\infty, 3s + i)\}, i \in [1, s - 1], (\text{null if } s = 1), \\ \{(0, 0), (3, 6s + 3 + 2i), (\infty, 9s + 3 + i)\}, i \in [0, s - 2], (\text{null if } s = 1). \end{array}$$

(2) $w = 12s + 8$. All $112s + 74$ are divided into two parts. The first part consists of following 11 base blocks:

$$\begin{array}{lll} \{(1, 0), (6, 0), (3, w/2)\}, & \{(0, 0), (1, 0), (2, 0)\}, & \{(0, 0), (5, w/2), (6, w/2)\}, \\ \{(0, 0), (5, 0), (1, w/2)\}, & \{(0, 0), (6, 0), (2, w/2)\}, & \{(0, 0), (3, w/2), (4, w/2)\}, \\ \{(1, 0), (3, 0), (2, w/2)\}, & \{(4, 0), (5, 0), (6, w/2)\}, & \{(1, 0), (4, w/2), (6, w/2)\}, \\ \{(2, 0), (3, 0), (4, w/2)\}, & \{(2, 0), (4, 0), (5, w/2)\}. \end{array}$$

The second part consists of $112s + 63$ base blocks. All base blocks can be obtained by developing following $16s + 9$ initial base blocks by $(+1, -) \bmod (7, -)$, where $\infty + 1 = \infty$.
 $s = 0$.

$$\begin{array}{lll} \{(0, 0), (\infty, 0), (2, 1)\}, & \{(0, 0), (3, 0), (1, 1)\}, & \{(0, 0), (3, 1), (2, 2)\}, \\ \{(0, 0), (4, 1), (\infty, 2)\}, & \{(0, 0), (3, 2), (4, 5)\}, & \{(0, 0), (5, 2), (2, 5)\}, \\ \{(0, 0), (4, 2), (\infty, 6)\}, & \{(0, 0), (1, 2), (\infty, 5)\}, & \{(0, 0), (6, 2), (1, 5)\}. \end{array}$$

$s \geq 1$.

$$\begin{aligned}
& \{(0, 0), (1, s+1), (\infty, 2s+2)\}, \{(0, 0), (1, 5s+3), (\infty, 11s+7)\}, \\
& \{(0, 0), (1, 4s+2), (3, 8s+4)\}, \{(0, 0), (4, 2s+1), (\infty, 9s+6)\}, \\
& \{(0, 0), (4, 4s+3), (\infty, 10s+6)\}, \{(0, 0), (\infty, s), (3, 10s+6)\}, \\
& \{(0, 0), (2, 8s+6+i), (4, 6s+5+2i)\}, i \in [0, 2s], \\
& \{(0, 0), (3, 2s+4+2i), (\infty, 3s+3+i)\}, i \in [0, s-1], \\
& \{(0, 0), (3, 4s+4+2i), (\infty, 6s+5+i)\}, i \in [0, s-1], \\
& \{(0, 0), (3, 6s+5+2i), (\infty, 5s+3+i)\}, i \in [0, s-1], \\
& \{(0, 0), (3, 8s+7+2i), (\infty, 8s+6+i)\}, i \in [0, s-1], \\
& \{(0, 0), (3, 8s+6+2i), (\infty, 4s+3+i)\}, i \in [0, s-1], \\
& \{(0, 0), (1, 10s+7-i), (3, 10s+8+i)\}, i \in [0, 4s+2], \\
& \{(0, 0), (1, 2s+2+i), (2, 2i+2)\}, i \in [0, 4s+1] \setminus \{2s, 3s+1\}, \\
& \{(0, 0), (3, 6s+6+2i), (\infty, 9s+7+i)\}, i \in [0, s-2], (\text{null if } s=1).
\end{aligned}$$

Lemma C.6 *There exists a 3-SCGDP of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks for any $w \equiv 10 \pmod{12}$.*

Proof Let $w = 12s + 10$. The required designs are constructed on $I \times Z_w$ with group set $\{\{i\} \times Z_w : i \in I\}$, where $I = Z_7 \cup \{\infty\}$. The $112s + 92$ base blocks are divided into two parts. The first part consists of 8 base blocks. In the second part, we only list $16s + 12$ initial base blocks. All other base blocks are obtained by developing the initial base blocks by $(+1, -) \pmod{(7, -)}$, where $\infty + 1 = \infty$.

Part I:

$$\begin{aligned}
& \{(2, 0), (3, 0), (4, 0)\}, \{(1, 0), (3, 0), (6, w/2)\}, \{(0, 0), (3, w/2), (5, 0)\}, \\
& \{(4, 0), (5, 0), (6, 0)\}, \{(0, 0), (2, w/2), (6, 0)\}, \{(1, 0), (4, w/2), (6, 0)\}, \\
& \{(0, 0), (1, 0), (2, 0)\}, \{(1, 0), (3, w/2), (5, w/2)\}.
\end{aligned}$$

Part II:

$$\begin{aligned}
& \{(0, 0), (3, 6s+4), (\infty, 7s+5)\}, \{(0, 0), (2, 8s+7), (\infty, s)\}, \\
& \{(0, 0), (1, 6s+5), (3, 10s+8)\}, \{(0, 0), (1, s+1), (3, 10s+9)\}, \\
& \{(0, 0), (1, 4s+3), (\infty, 9s+7)\}, \{(0, 0), (1, 5s+4), (3, 4s+3)\}, \\
& \{(0, 0), (3, 8s+7+2i), (\infty, 8s+6+i)\}, i \in [0, s], \\
& \{(0, 0), (3, 6s+6+2i), (\infty, 9s+8+i)\}, i \in [0, s], \\
& \{(0, 0), (2, 8s+8+i), (4, 6s+7+2i)\}, i \in [0, 2s] \setminus \{s\}, \\
& \{(0, 0), (1, 10s+8-i), (3, 10s+10+i)\}, i \in [0, 4s+2], \\
& \{(0, 0), (1, 2s+2+i), (2, 1+2i)\}, i \in [0, 4s+2] \setminus \{2s+1, 3s+2\}, \\
& \{(0, 0), (3, 2s+4+2i), (\infty, 3s+3+i)\}, i \in [0, s-1] \text{ (null if } s=0), \\
& \{(0, 0), (3, 4s+4+2i), (\infty, 6s+5+i)\}, i \in [0, s-1] \text{ (null if } s=0), \\
& \{(0, 0), (3, 6s+7+2i), (\infty, 5s+5+i)\}, i \in [0, s-1] \text{ (null if } s=0), \\
& \{(0, 0), (3, 8s+8+2i), (\infty, 4s+3+i)\}, i \in [0, s-1] \text{ (null if } s=0).
\end{aligned}$$

Lemma C.7 *There exists a 3-SCGDP of type w^8 with $J^*(8 \times 1 \times w, 3, 1)$ base blocks for any $w \equiv 5 \pmod{6}$.*

Proof Let $w = 6s + 5$. The required designs are constructed on $I \times Z_w$ with group set $\{\{i\} \times Z_w : i \in I\}$, where $I = Z_7 \cup \{\infty\}$. The $56s + 45$ base blocks are divided into two parts. The first part consists of 3 base blocks. In the second part, we only list $8s + 6$ initial base blocks. All other base blocks are obtained by developing the initial base blocks by $(+1, -) \pmod{(7, -)}$, where $\infty + 1 = \infty$.

• $s \in \{0, 1, 2\}$:

Part I:

$$\{(0, 0), (1, 0), (\infty, 0)\}, \{(2, 0), (3, 0), (\infty, 0)\}, \{(4, 0), (5, 0), (\infty, 0)\}.$$

Part II:

$s = 0$:

$$\{(0, 0), (2, 0), (1, 1)\}, \{(0, 0), (3, 0), (5, 2)\}, \{(0, 0), (2, 1), (\infty, 2)\}, \\ \{(0, 0), (4, 1), (1, 3)\}, \{(0, 0), (3, 1), (6, 3)\}, \{(0, 0), (5, 1), (\infty, 4)\}.$$

$s = 1$:

$$\{(0, 0), (4, 3), (1, 8)\}, \{(0, 0), (1, 4), (\infty, 8)\}, \{(0, 0), (2, 0), (1, 1)\}, \{(0, 0), (3, 0), (5, 5)\}, \\ \{(0, 0), (2, 1), (6, 2)\}, \{(0, 0), (\infty, 3), (3, 7)\}, \{(0, 0), (5, 1), (2, 9)\}, \{(0, 0), (\infty, 1), (1, 7)\}, \\ \{(0, 0), (1, 2), (4, 7)\}, \{(0, 0), (2, 2), (4, 9)\}, \{(0, 0), (4, 2), (5, 8)\}, \{(0, 0), (\infty, 2), (5, 7)\}, \\ \{(0, 0), (1, 3), (2, 8)\}, \{(0, 0), (3, 1), (\infty, 10)\}.$$

$s = 2$:

$$\{(0, 0), (\infty, 5), (3, 9)\}, \{(0, 0), (3, 4), (2, 8)\}, \{(0, 0), (2, 0), (1, 1)\}, \{(0, 0), (4, 1), (\infty, 16)\}, \\ \{(0, 0), (2, 1), (5, 14)\}, \{(0, 0), (3, 0), (4, 5)\}, \{(0, 0), (5, 1), (1, 12)\}, \{(0, 0), (\infty, 1), (6, 8)\}, \\ \{(0, 0), (1, 2), (4, 7)\}, \{(0, 0), (3, 1), (1, 11)\}, \{(0, 0), (2, 2), (5, 5)\}, \{(0, 0), (3, 2), (\infty, 14)\}, \\ \{(0, 0), (4, 2), (5, 6)\}, \{(0, 0), (5, 2), (2, 13)\}, \{(0, 0), (6, 2), (\infty, 8)\}, \{(0, 0), (\infty, 2), (5, 8)\}, \\ \{(0, 0), (1, 3), (2, 10)\}, \{(0, 0), (4, 3), (5, 11)\}, \{(0, 0), (\infty, 3), (6, 11)\}, \{(0, 0), (6, 3), (\infty, 7)\}, \\ \{(0, 0), (5, 3), (4, 10)\}, \{(0, 0), (2, 4), (4, 9)\}.$$

• $s \equiv 1 \pmod{2}$ and $s \geq 3$:

Part I:

$$\{(4, 0), (5, 0), (\infty, (11s+9)/2)\}, \{(0, 0), (1, 0), (\infty, (11s+9)/2)\}, \{(2, 0), (3, 0), (\infty, (11s+9)/2)\}.$$

Part II:

$$\{(0, 0), (3, 4s+2), (\infty, 4s+2)\}, \{(0, 0), (3, s+1), (\infty, 5s+4)\}, \\ \{(0, 0), (3, 3s+2), (\infty, 5s+3)\}, \{(0, 0), (1, 2s+2), (3, 4s+4)\}, \\ \{(0, 0), (3, 4s+3), (\infty, (5s+3)/2)\}, \\ \{(0, 0), (1, 5s+3-i), (3, 5s+4+i)\}, i \in [0, 2s], \\ \{(0, 0), (2, 4s+3+i), (4, 3s+2+2i)\}, i \in [0, s], \\ \{(0, 0), (1, s+1+i), (2, 2i)\}, i \in [0, 2s+1] \setminus \{s+1\}, \\ \{(0, 0), (3, s+2i), (\infty, (3s+1)/2+i)\}, i \in [0, (s-1)/2], \\ \{(0, 0), (3, 2s+1+2i), (\infty, 3s+2+i)\}, i \in [0, (s-1)/2], \\ \{(0, 0), (3, 4s+6+2i), (\infty, 2s+2+i)\}, i \in [0, (s-3)/2], \\ \{(0, 0), (3, 4s+5+2i), (\infty, 4s+4+i)\}, i \in [0, (s-3)/2], \\ \{(0, 0), (3, 3s+4+2i), (\infty, (5s+5)/2+i)\}, i \in [0, (s-3)/2], \\ \{(0, 0), (3, 3s+5+2i), (\infty, (9s+9)/2+i)\}, i \in [0, (s-5)/2] \text{ (null if } s=3).$$

• $s \equiv 0 \pmod{2}$ and $s \geq 4$:

Part I:

$$\{(0, 0), (1, 0), (\infty, 11s/2+5)\}, \{(2, 0), (3, 0), (\infty, 11s/2+5)\}, \{(4, 0), (5, 0), (\infty, 11s/2+5)\}.$$

Part II:

$$\begin{aligned}
& \{(0,0), (1,2s+1), (2,2s-2)\}, \{(0,0), (1,s+1), (2,1)\}, \\
& \{(0,0), (1,5s+3), (3,5s+3)\}, \{(0,0), (1,2s), (3,4s)\}, \\
& \{(0,0), (3,s+1), (\infty,3s+3)\}, \{(0,0), (3,4s+2), (\infty,0)\}, \\
& \{(0,0), (1,2s+2), (3,4s+4)\}, \{(0,0), (1,6s+4), (\infty,5s+2)\}, \\
& \{(0,0), (3,5s+4), (\infty,9s/2+3)\}, \\
& \{(0,0), (2,4s+3+i), (4,3s+2+2i)\}, i \in [0, s], \\
& \{(0,0), (3,s+2i), (\infty,3s/2+1+i)\}, i \in [0, s/2], \\
& \{(0,0), (1,s+3+2i), (2,2+4i)\}, i \in [0, s/2-2], \\
& \{(0,0), (1,s+2+2i), (2,4+4i)\}, i \in [0, s/2-2], \\
& \{(0,0), (3,2s+2+2i), (\infty,3s+4+i)\}, i \in [0, s/2], \\
& \{(0,0), (1,2s+3+i), (2,2s+4+2i)\}, i \in [0, s-1], \\
& \{(0,0), (1,5s+2-i), (3,5s+5+i)\}, i \in [0, 2s-1], \\
& \{(0,0), (3,4s+6+2i), (\infty,2s+4+i)\}, i \in [0, s/2-2], \\
& \{(0,0), (3,4s+5+2i), (\infty,4s+4+i)\}, i \in [0, s/2-2], \\
& \{(0,0), (3,3s+5+2i), (\infty,5s/2+3+i)\}, i \in [0, s/2-1], \\
& \{(0,0), (3,3s+4+2i), (\infty,9s/2+4+i)\}, i \in [0, s/2-3] \text{ (null if } s=4).
\end{aligned}$$

Lemma C.8 *There exists a 3-SCGDP of type w^{14} with $J^*(14 \times 1 \times w, 3, 1)$ base blocks for any $w \equiv 5 \pmod{6}$.*

Proof When $w = 5$, the required design is constructed on $I \times Z_5$ with group set $\{\{i\} \times Z_5 : i \in I\}$, where $I = Z_{13} \cup \{\infty\}$. The 149 base blocks are divided into two parts. The first part consists of 6 base blocks.

$$\begin{aligned}
& \{(0,0), (1,0), (\infty,0)\}, \quad \{(2,0), (3,0), (\infty,0)\}, \quad \{(4,0), (5,0), (\infty,0)\}, \\
& \{(6,0), (7,0), (\infty,0)\}, \quad \{(8,0), (9,0), (\infty,0)\}, \quad \{(10,0), (11,0), (\infty,0)\}.
\end{aligned}$$

In the second part, we only list 11 initial base blocks. All base blocks are obtained by developing the initial base blocks by $(+1, -)$ modulo $(13, -)$, where $\infty + 1 = \infty$.

$$\begin{aligned}
& \{(0,0), (8,1), (2,3)\}, \quad \{(0,0), (2,0), (5,0)\}, \quad \{(0,0), (4,1), (3,2)\}, \quad \{(0,0), (4,2), (\infty,4)\}, \\
& \{(0,0), (4,0), (1,1)\}, \quad \{(0,0), (6,0), (2,1)\}, \quad \{(0,0), (3,1), (1,2)\}, \quad \{(0,0), (\infty,1), (8,3)\}, \\
& \{(0,0), (5,1), (7,3)\}, \quad \{(0,0), (6,1), (1,3)\}, \quad \{(0,0), (7,1), (3,3)\}.
\end{aligned}$$

When $w \geq 11$, let $X = I_{14} \times Z_w$ and $\mathcal{G} = \{\{i\} \times Z_w : i \in I_{14}\}$. We divide the construction of the $J^*(14 \times 1 \times w, 3, 1) = (91w - 8)/3$ base blocks into three parts.

Part I: Let (I_{14}, \mathcal{B}) be a PBD $(14, \{3, 4, 5^*\})$ and I_5 be the unique block of size 5. For each $B \in \mathcal{B} \setminus \{I_5\}$, construct a 3-SCHGDD of type $(|B|, 1^w)$ on $B \times Z_w$ with group set $\{\{x\} \times Z_w : x \in B\}$ and hole set $\{B \times \{i\} : i \in Z_w\}$. Denote the base blocks by \mathcal{A}_B . Let $\mathcal{A}_1 = \sum_{B \in \mathcal{B} \setminus \{I_5\}} \mathcal{A}_B$. It is readily checked that $|\mathcal{A}_1| = ((\binom{14}{2} - \binom{5}{2}))(w-1)/3 = 27(w-1)$.

Part II: Construct a 3-GDP of type 1^{14} on $I_{14} \times \{0\}$, which consists of the following 26 blocks. Let \mathcal{A}_2 denote the block set. For saving space, we omit the second candidates.

$$\begin{aligned}
& \{0, 8, 10\}, \{0, 9, 12\}, \{2, 7, 11\}, \{0, 4, 6\}, \{6, 9, 10\}, \{3, 6, 8\}, \\
& \{1, 6, 11\}, \{1, 7, 12\}, \{2, 5, 10\}, \{1, 2, 8\}, \{2, 6, 12\}, \{3, 10, 11\}, \\
& \{3, 4, 13\}, \{3, 5, 12\}, \{4, 7, 10\}, \{3, 7, 9\}, \{7, 8, 13\}, \{1, 10, 13\}, \\
& \{4, 8, 12\}, \{4, 9, 11\}, \{5, 6, 13\}, \{1, 5, 9\}, \{5, 8, 11\}, \{0, 11, 13\}, \\
& \{2, 9, 13\}, \{0, 5, 7\}.
\end{aligned}$$

Let $M = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 12\}, \{11, 12\}, \{12, 13\}\}$. Note that all pairs in M are not covered by above blocks.

Part III: Construct a 3-SCGDP of type w^5 on $I_5 \times Z_w$ with group set $\{\{i\} \times Z_w : i \in I_5\}$, which consists of $(10w - 5)/3$ base blocks. Let \mathcal{A}_3 denote these base blocks.

We first list $2(w - 8)/3$ initial base blocks. We can obtain $10(w - 8)/3$ base blocks by developing these blocks by $(+1, -)$ modulo $(5, -)$. Here, $2 \leq i \leq (w - 5)/3$.

$$\{(0, 0), (1, i), (2, 2(w - 2)/3 + 2i)\}, \{(0, 0), (1, (w - 2)/3 + i), (3, 2(w - 2)/3 - i + 1)\}.$$

The remaining 25 base blocks in this part are listed below.

$$\begin{aligned} &\{(1, 0), (2, 1), (4, (w + 1)/3)\}, & \{(0, 0), (1, 1), (3, (w - 5)/3)\}, \\ &\{(0, 0), (4, 1), (1, (w - 2)/3)\}, & \{(0, 0), (4, 2), (2, (w + 1)/3)\}, \\ &\{(1, 0), (4, (w - 2)/3), (2, w - 2)\}, & \{(0, 0), (3, 2(w + 1)/3), (1, w - 1)\}, \\ &\{(0, 0), (4, 2(w + 1)/3), (1, w - 2)\}, & \{(0, 0), (2, (2w + 5)/3), (4, w - 1)\}, \\ &\{(0, 0), (2, (w - 5)/3), (3, (w - 2)/3)\}, & \{(0, 0), (2, (w - 2)/3), (1, (w + 1)/3)\}, \\ &\{(0, 0), (2, (w - 8)/3), (1, 2(w - 2)/3)\}, & \{(0, 0), (3, (w + 1)/3), (1, (2w - 1)/3)\}, \\ &\{(0, 0), (4, (w + 1)/3), (2, (2w - 1)/3)\}, & \{(1, 0), (3, (w - 5)/3), (2, (w + 1)/3)\}, \\ &\{(1, 0), (4, (w - 5)/3), (3, (w + 1)/3)\}, & \{(1, 0), (2, (w - 2)/3), (4, (2w - 1)/3)\}, \\ &\{(1, 0), (3, (w - 2)/3), (2, (2w - 1)/3)\}, & \{(1, 0), (3, (w + 4)/3), (4, 2(w + 1)/3)\}, \\ &\{(2, 0), (4, (w - 5)/3), (3, 2(w - 2)/3)\}, & \{(2, 0), (3, (w - 2)/3), (4, (2w - 1)/3)\}, \\ &\{(0, 0), (2, (w + 4)/3), (3, (2w + 5)/3)\}, & \{(0, 0), (4, (w + 4)/3), (3, 2(w + 4)/3)\}, \\ &\{(1, 0), (4, 2(w - 2)/3), (3, (2w - 1)/3)\}, & \{(0, 0), (3, 2(w - 2)/3), (4, (2w - 1)/3)\}, \\ &\{(0, 0), (3, (2w - 1)/3), (2, 2(w + 1)/3)\}. \end{aligned}$$

Finally, let $\mathcal{A} = \sum_{i=1}^3 \mathcal{A}_i$. It is readily checked that \mathcal{A} forms the base blocks of the required design. Clearly $|\mathcal{A}| = (91w - 8)/3$. \square